Criteriul lui Cauchy d'Alembert

i) Considerații teoretice

$$\left.\begin{array}{l}(x_n), x_n > 0 \\ \text{Dacă } \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \alpha\end{array}\right| \Rightarrow \exists \lim\limits_{n\to\infty} \sqrt[n]{x_n} = \alpha$$

ii) Exerciții rezolvate

1. Să se calculeze $\lim\limits_{n\to\infty} \sqrt[n]{n!\cdot n}$

Rezolvare:

$$\text{Fie } x_n = n!\cdot n \Rightarrow \frac{x_{n+1}}{x_n} = \frac{(n+1)!\cdot(n+1)}{n!\cdot n} = \frac{\cancel{n!}\,(n+1)\cdot(n+1)}{\cancel{n!}\cdot n} = \frac{(n+1)^2}{2}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2}{n} = \infty \overset{\text{C.d'A}}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{n!\cdot n} = \infty .$$

2. Să se calculeze $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n!}{n+1}}$

Rezolvare:

$$\text{Fie } x_n = \frac{n!}{n+1} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{\dfrac{(n+1)!}{n+2}}{\dfrac{n!}{n+1}} = \frac{\cancel{n!}\,(n+1)}{n+2} \cdot \frac{n+1}{\cancel{n!}} = \frac{(n+1)^2}{n+2}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2}{n+2} = \infty \overset{\text{C.d'A}}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n+1}} = \infty .$$

3. Să se calculeze $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{(n+1)(n+2)\ldots(2n)}{n!}}$

Rezolvare:

$$\text{Fie } x_n = \frac{(n+1)(n+2)\ldots(2n)}{n!} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{\dfrac{(n+2)(n+3)\ldots(2n)(2n+1)(n+2)}{(n+1)!}}{\dfrac{(n+1)(n+2)\ldots(2n)}{n!}} =$$

$$= \frac{\cancel{(n+2)(n+3)\ldots(2n)}\,(2n+1)(2n+2)}{\cancel{n!}\,(n+1)} \cdot \frac{\cancel{n!}}{(n+1)\,\cancel{(n+2)\ldots(2n)}} = \frac{(2n+1)\cdot 2\,\cancel{(n+1)}}{\cancel{(n+1)}\,(n+1)} =$$

$$= \frac{4n+2}{n+1}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{4n+2}{n+1} = 4 \overset{\text{C.d'A}}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{\frac{(n+1)(n+2)\ldots(2n)}{n!}} = 4 .$$

4. Să se calculeze $\lim_{n\to\infty}\sqrt[n]{\frac{n\ln n}{n^{\ln n}}}$

Rezolvare:

$$\text{Fie } x_n=\frac{n\ln n}{n^{\ln n}}\Rightarrow\frac{x_{n+1}}{x_n}=\frac{\frac{(n+1)\ln(n+1)}{(n+1)^{\ln(n+1)}}}{\frac{n\ln n}{n^{\ln n}}}=\frac{(n+1)\ln(n+1)}{(n+1)^{\ln(n+1)}}\cdot\frac{n^{\ln n}}{n\ln n}=$$

$$=\underbrace{\frac{n+1}{n}}_{\to1}\cdot\underbrace{\frac{\ln(n+1)}{\ln n}}_{\to1}\cdot\frac{n^{\ln n}}{(n+1)^{\ln(n+1)}}=\underbrace{\frac{n+1}{n}\cdot\frac{\ln(n+1)}{\ln n}}_{A}\cdot\frac{n^{\ln n}}{(n+1)^{\ln n\left(1+\frac{1}{n}\right)}}=$$

$$=A\cdot\frac{n^{\ln n}}{(n+1)^{\ln n+\ln\left(1+\frac{1}{n}\right)}}=A\cdot\frac{n^{\ln n}}{(n+1)^{\ln n}}\cdot\frac{1}{(n+1)^{\ln\left(1+\frac{1}{n}\right)}}=A\cdot\left(\frac{n}{n+1}\right)^{\ln n}\cdot\frac{1}{(n+1)^{\ln\left(1+\frac{1}{n}\right)}}=$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{\ln n}=\lim_{n\to\infty}\left(1+\frac{n}{n+1}-1\right)^{\ln n}=\lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-\ln n}{n+1}}=e^{-\lim\limits_{n\to\infty}\underbrace{\frac{\ln n}{n+1}}_{0}}=1$$

$$\lim_{n\to\infty}(n+1)^{\ln\left(1+\frac{1}{n}\right)}=\lim_{n\to\infty}e^{\ln\left[(n+1)^{\ln\left(1+\frac{1}{n}\right)}\right]}=\lim_{n\to\infty}e^{\ln\left(1+\frac{1}{n}\right)\ln(n+1)}=\lim_{n\to\infty}e^{\overbrace{\frac{\ln(n+1)}{n}}^{0}\cdot n\ln\left(1+\frac{1}{n}\right)}=$$

$$=\lim_{n\to\infty}e^{\overbrace{\frac{\ln(n+1)}{n}}^{0}\cdot\overbrace{\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}}^{1}}=1$$

Prin urmare,

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=1\overset{C.d'A}{\Rightarrow}\lim_{n\to\infty}\sqrt[n]{\frac{n\ln n}{n^{\ln n}}}=1.$$

5. Să se calculeze $\lim_{n\to\infty}\sqrt[n]{\frac{[(n+1)!]^2}{(2n)!\cdot5^n}}$

Rezolvare:

$$\text{Fie } x_n=\frac{[(n+1)!]^2}{(2n)!\cdot5^n}\Rightarrow\frac{x_{n+1}}{x_n}=\frac{\frac{[(n+2)!]^2}{[2(n+1)]!\cdot5^{n+1}}}{\frac{[(n+1)!]^2}{(2n)!\cdot5^n}}=\frac{[(n+2)!]^2}{[2(n+1)]!\cdot5^{n+1}}\cdot\frac{(2n)!\cdot5^n}{[(n+1)!]^2}=$$

$$=\frac{\left[\cancel{(n+1)!}\,(n+2)\right]^2}{\cancel{(2n)!}\,(2n+1)(2n+2)\cdot\cancel{5^n}\cdot5}\cdot\frac{\cancel{(2n)!}\cdot\cancel{5^n}}{\left[\cancel{(n+1)!}\right]^2}=\frac{(n+2)^2}{5(2n+1)(2n+2)}$$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{(n+2)^2}{5(2n+1)(2n+2)}=\lim_{n\to\infty}\frac{\cancel{n^2}\left(1+\frac{2}{n}\right)^2}{20\cancel{n^2}\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}=\frac{1}{20}\Rightarrow$$

$$\overset{C.d'A}{\Rightarrow}\lim_{n\to\infty}\sqrt[n]{\frac{\left[(n+1)!\right]^2}{(2n)!\cdot 5^n}}=\frac{1}{20}\ .$$