Criteriul raportului

i) Considerații teoretice

$$\left.\begin{array}{l}(x_n), x_n > 0 \\ \text{Dacă } \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \alpha > 0\end{array}\right| \Rightarrow \begin{cases} 1.\text{dacă } \alpha \in (0,1) \Rightarrow \lim\limits_{n\to\infty} x_n = 0 \\ 2.\text{dacă } \alpha > 1 \Rightarrow \lim\limits_{n\to\infty} x_n = \infty \\ 3.\text{dacă } \alpha = 1 \Rightarrow \text{criteriu ineficient} \end{cases}$$

ii) Exerciții rezolvate

1. Să se calculeze $\lim\limits_{n\to\infty} \dfrac{n^3}{2^n}$

Rezolvare:

$$\text{Fie } x_n = \frac{n^3}{2^n} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{\dfrac{(n+1)^3}{2^{n+1}}}{\dfrac{n^3}{2^n}} = \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} = \left(\frac{n+1}{n}\right)^3 \cdot \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^3 = \frac{1}{2} > 0 \overset{\text{C.Rap.}}{\Rightarrow} \lim_{n\to\infty} \frac{n^3}{2^n} = \frac{1}{2}.$$

2. Să se calculeze $\lim\limits_{n\to\infty} \dfrac{3^n}{n!}$

Rezolvare:

$$\text{Fie } x_n = \frac{3^n}{n!} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{\dfrac{3^{n+1}}{(n+1)!}}{\dfrac{3^n}{n!}} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{\cancel{3^n} \cdot 3}{\cancel{n!}(n+1)} \cdot \frac{\cancel{n!}}{\cancel{3^n}} = \frac{3}{n+1}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 \overset{\text{C.Rap.}}{\Rightarrow} \lim_{n\to\infty} \frac{3^n}{n!} = 0 \text{ .abbb}$$

3. Să se calculeze $\lim\limits_{n\to\infty} \dfrac{5^n \cdot n!}{2^n \cdot n^n}$

Rezolvare:

$$\text{Fie } x_n = \frac{5^n \cdot n!}{2^n \cdot n^n} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{\dfrac{5^{n+1} \cdot (n+1)!}{2^{n+1} \cdot (n+1)^{n+1}}}{\dfrac{5^n \cdot n!}{2^n \cdot n^n}} = \frac{\cancel{5^n} \cdot 5 \cdot \cancel{n!}(n+1)}{\cancel{2^n} \cdot 2 \cdot (n+1)^{n+1}} \cdot \frac{\cancel{2^n} \cdot n^n}{\cancel{5^n} \cdot \cancel{n!}} =$$

$$= \frac{5 \cdot (n+1) n^n}{2 \cdot (n+1)^{n+1}} = \frac{5n^n}{2(n+1)^n} = \frac{5}{2}\left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\left(1+\frac{n}{n+1}-1\right)^n=\lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}}=e^{-\lim\limits_{n\to\infty}\frac{n}{n+1}}=\frac{1}{e}$$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{5}{2}\underbrace{\left(\frac{n}{n+1}\right)^n}_{1}=\frac{5}{2}\overset{\text{C.Rap.}}{\Rightarrow}\lim_{n\to\infty}\frac{5^n\cdot n!}{2^n\cdot n^n}=\frac{5}{2}.$$

iii) Exerciții propuse

Să se calculeze următoarele limitele:

i) $\lim\limits_{n\to\infty}\dfrac{n!\cdot(2n)!}{(3n)!}$;

ii) $\lim\limits_{n\to\infty}\dfrac{(n!)^2}{(2n)!\cdot 8^n}$;

iii) $\lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)\cdot\ldots\cdot(2n)}{n!}$;

iv) $\lim\limits_{n\to\infty}\dfrac{n(n+1)(n+2)\cdot\ldots\cdot(2n)}{n^n}$.