Limite remarcabile-Funcții trigonometrice

i) Considerații teoretice

Aceste limite elimină o nedeterminare de forma $\frac{0}{0}$

$$\boxed{\begin{array}{l}\lim\limits_{x\to 0}\frac{\sin x}{x}=1;\\ \lim\limits_{x\to 0}\frac{tgx}{x}=1;\\ \lim\limits_{x\to 0}\frac{\arcsin x}{x}=1;\\ \lim\limits_{x\to 0}\frac{arctgx}{x}=1\end{array}} \text{ și } \boxed{\begin{array}{l}\lim\limits_{u(x)\to 0}\frac{\sin u(x)}{u(x)}=1;\\ \lim\limits_{u(x)\to 0}\frac{tgu(x)}{u(x)}=1;\\ \lim\limits_{u(x)\to 0}\frac{\arcsin u(x)}{u(x)}=1;\\ \lim\limits_{u(x)\to 0}\frac{arctgu(x)}{u(x)}=1\end{array}}$$

ii) Exerciții rezolvate

1. Să se calculeze limitele:

i) $\lim\limits_{x\to 0}\frac{\sin 5x}{10x}$;

ii) $\lim\limits_{x\to 0}\frac{tg2x}{\arcsin 6x}$;

iii) $\lim\limits_{x\to\infty}(x+2)\sin\frac{1}{x+2}$;

iv) $\lim\limits_{x\to\infty}\left(x^2+x+1\right)\arcsin\frac{1}{2\left(x^2+x+1\right)}$;

v) $\lim\limits_{x\to 0}\frac{\arcsin 2x+\arcsin 4x}{\arcsin 8x}$.

Rezolvare:

i) $$\lim\limits_{x\to 0}\frac{\sin 5x}{10x}=\frac{\sin 0}{0}\overset{\left[\frac{0}{0}\right]}{=}\lim\limits_{x\to 0}\frac{\overbrace{\frac{\sin 5x}{5x}}^{1}\cdot 5x}{10x}\lim\limits_{x\to 0}\frac{5x}{10x}=\frac{5}{10}=\frac{1}{2};$$

ii) $$\lim\limits_{x\to 0}\frac{tg2x}{\arcsin 6x}=\frac{tg0}{\arcsin 0}\overset{\left[\frac{0}{0}\right]}{=}\lim\limits_{x\to 0}\frac{\overbrace{\frac{tg2x}{2x}}^{1}\cdot 2x}{\underbrace{\frac{\arcsin 6x}{6x}}_{1}\cdot 6x}=\lim\limits_{x\to 0}\frac{2x}{6x}=\frac{2}{6}=\frac{1}{3};$$

iii) $\lim\limits_{x\to\infty}(x+2)\sin\dfrac{1}{x+2}=\infty\cdot 0=\dfrac{0}{\frac{1}{\infty}}=\left[\dfrac{0}{0}\right]$;

Notăm $t=\dfrac{1}{x+2}$; Când $x\to\infty\Rightarrow t\to 0$

Limita noastră devine $\lim\limits_{t\to 0}\dfrac{1}{t}\sin t=\lim\limits_{t\to 0}\dfrac{1}{t}\sin t=\lim\limits_{t\to 0}\dfrac{\sin t}{t}=1$

iv) $\lim\limits_{x\to\infty}\left(x^2+x+1\right)\arcsin\dfrac{1}{2\left(x^2+x+1\right)}=\infty\cdot 0=\dfrac{0}{\frac{1}{\infty}}=\left[\dfrac{0}{0}\right]$

Notăm $t=\dfrac{1}{x^2+x+1}$; Când $x\to\infty\Rightarrow t\to 0$

Limita noastră devine $\lim\limits_{t\to 0}\dfrac{1}{t}\arcsin\dfrac{t}{2}=\lim\limits_{t\to 0}\dfrac{1}{t}\underbrace{\dfrac{\arcsin\frac{t}{2}}{\frac{t}{2}}}_{1}\cdot\dfrac{t}{2}=\lim\limits_{t\to 0}\dfrac{1}{t}\cdot\dfrac{t}{2}=\dfrac{1}{2}$

v) $\lim\limits_{x\to 0}\dfrac{\arcsin 2x+\arcsin 4x}{\arcsin 8x}=\dfrac{\arcsin 0}{\arcsin 0}\overset{\left[\frac{0}{0}\right]}{=}\lim\limits_{x\to 0}\dfrac{\overbrace{\dfrac{\arcsin 2x}{2x}}^{1}\cdot 2x+\overbrace{\dfrac{\arcsin 4x}{4x}}^{1}\cdot 4x}{\underbrace{\dfrac{\arcsin 8x}{8x}}_{1}\cdot 8x}=\lim\limits_{x\to 0}\dfrac{6x}{8x}=\dfrac{3}{4}$.

2. Să se calculeze limitele:

i) $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x^2-1}$;

ii) $\lim\limits_{x\to -3}\dfrac{x^2+4x+3}{\arcsin^2(x+3)}$;

iii) $\lim\limits_{x\to 2}\dfrac{\mathrm{tg}\left(x^2-4\right)}{\sin\left(x^2-5x+6\right)}$.

Rezolvare:

i) $L=\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x^2-1}=\dfrac{\sin 0}{0}\overset{\left[\frac{0}{0}\right]}{=}\lim\limits_{x\to 1}\dfrac{\overbrace{\dfrac{\sin(x-1)}{x-1}}^{1}\cdot(x-1)}{x^2-1}=\lim\limits_{x\to 1}\dfrac{x-1}{x^2-1}=\dfrac{1-1}{1^2-1}=\dfrac{0}{0}$

$\boxed{(a-b)(a+b)=a^2-b^2}$

Aplicând formula de mai sus avem $x^2-1=(x-1)(x+1)$

$L=\lim\limits_{x\to 1}\dfrac{x-1}{x^2-1}=\lim\limits_{x\to 1}\dfrac{\cancel{x-1}}{(\cancel{x-1})(x+1)}=\lim\limits_{x\to 1}\dfrac{1}{x+1}=\dfrac{1}{2}$

$$\text{ii) } L=\lim_{x\to -3}\frac{x^2+4x+3}{\arcsin^2(x+3)}=\frac{0}{\arcsin 0}\overset{\left[\frac{0}{0}\right]}{=}\lim_{x\to -3}\frac{x^2+4x+3}{\underbrace{\frac{\arcsin(x+3)}{x+3}}_{1}(x+3)\cdot\underbrace{\frac{\arcsin(x+3)}{x+3}}_{1}(x+3)}=$$

$$=\lim_{x\to -3}\frac{x^2+4x+3}{(x+3)\cdot(x+3)}=\frac{(-3)^2+4\cdot(-3)+3}{(-3+3)(-3+3)}=\frac{0}{0}.$$

$$\boxed{ax^2+bx+c=a(x-x_1)(x-x_2)}$$

$x^2+4x+3=a(x-x_1)(x-x_2)=(x-x_1)(x-x_2)$

$x^2+4x+3=0, \Delta=4, x_1=-1, x_2=-3 \Rightarrow x^2+4x+3=(x+1)(x+3)$

$$L=\lim_{x\to -3}\frac{x^2+4x+3}{(x+3)\cdot(x+3)}=\lim_{x\to -3}\frac{(x+1)\cdot\cancel{(x+3)}}{(x+3)\cdot\cancel{(x+3)}}=\lim_{x\to -3}\frac{x+1}{x+3}=\frac{-2}{0}$$

Calculăm limitele laterale în -3

$$\left.\begin{array}{l} l_s(-3)=\lim\limits_{\substack{x\to -3\\ x<-3}}\frac{x+1}{x+3}=\frac{-2}{0_-}=+\infty \\ l_d(-3)=\lim\limits_{\substack{x\to -3\\ x>-3}}\frac{x+1}{x+3}=\frac{-2}{0_+}=-\infty \end{array}\right| \Rightarrow \nexists \lim_{x\to -3}\frac{x+1}{x+3} \Rightarrow \nexists \lim_{x\to -3}\frac{x^2+4x+3}{\arcsin^2(x+3)}$$

$$\text{iii) } \lim_{x\to 2}\frac{\operatorname{tg}(x^2-4)}{\sin(x^2-5x+6)}\overset{\left[\frac{0}{0}\right]}{=}\lim_{x\to 2}\frac{\operatorname{tg}(x^2-4)}{\sin(x^2-5x+6)}=\lim_{x\to 2}\frac{\overbrace{\frac{\operatorname{tg}(x^2-4)}{x^2-4}}^{1}(x^2-4)}{\underbrace{\frac{\sin(x^2-5x+6)}{x^2-5x+6}}_{1}(x^2-5x+6)}=$$

$$=\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}=\frac{2^2-4}{2^2-5\cdot 2+6}=\frac{0}{0}$$

Folosind formula $\boxed{(a-b)(a+b)=a^2-b^2}$ avem $x^2-4=(x-2)(x+2)$

$$\boxed{ax^2+bx+c=a(x-x_1)(x-x_2)}$$

$x^2-5x+6=a(x-x_1)(x-x_2)=(x-x_1)(x-x_2)$

$x^2-5x+6=0, \Delta=1, x_1=2, x_2=3 \Rightarrow x^2-5x+6=(x-2)(x-3)$

$$\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}=\lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-3)}=\lim_{x\to 2}\frac{x+2}{x-3}=-4.$$

iii) Exerciții propuse

1. Să se calculeze limitele:

i) $\lim\limits_{x\to 0}\frac{\sin 3x}{12x}$;   ii) $\lim\limits_{x\to 0}\frac{\operatorname{tg}5x}{\sin 25x}$;

iii) $\lim\limits_{x\to \infty}(2x+1)\sin\frac{1}{2x+1}$;   iv) $\lim\limits_{x\to \infty}\left(x^2+3\right)\arcsin\frac{1}{5\left(x^2+3\right)}$;

v) $\lim\limits_{x\to 0}\frac{\sin 3x+\sin 5x}{\arcsin 2x+\arcsin 4x}$;   vi) $\lim\limits_{x\to 0}\frac{\sin^3 x}{\sin x^3}$;

vii) $\lim\limits_{x\to 0}\frac{\sin^5 x}{\left(x^2+x\right)\sin x^4}$.

2. Să se calculeze limitele:

i) $\lim\limits_{x\to 2}\frac{\sin(x-2)}{x^2-4}$;   ii) $\lim\limits_{x\to 1}\frac{x^2+x-2}{\operatorname{tg}^2(x-1)}$;

iii) $\lim\limits_{x\to -3}\frac{\arcsin\left(x^2-9\right)}{\operatorname{arctg}\left(x^2+7x+12\right)}$;   iv) $\lim\limits_{x\to 1}\frac{\sin\left(x^2-5x+4\right)}{x^3-1}$;

v) $\lim\limits_{x\to \frac{1}{3}}\frac{\operatorname{arctg}(1-3x)}{\sin\left(9x^2-1\right)}$;   vi) $\lim\limits_{x\to \frac{2}{3}}\frac{\operatorname{tg}(3x-2)}{3x^2+x-2}$;

vii) $\lim\limits_{x\to 0}\frac{\sin(x\sin 2x)}{x^2}$;   viii) $\lim\limits_{x\to \frac{\pi}{2}}\frac{1-\sin x}{(\pi-2x)^2}$.