Proprietăți ale integralei definite

Fie $f, g : [a,b] \to \mathbb{R}$ funcții continue pe $[a,b]$

- $\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

- $\int_a^b \alpha \cdot f(x)\,dx = \alpha \cdot \int_a^b f(x)\,dx$

- $\int_a^b \left[\alpha f(x) + \beta g(x)\right] dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$

  proprietatea de liniaritate

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad \forall c \in [a,b]$

  proprietatea de aditivitate

- Dacă $f(x) \leq g(x), \forall x \in [a,b]$

  atunci $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

  proprietatea de monatonie.

Caz particular:

- Dacă $f(x) \geq 0, \ \forall x \in [a,b] \Rightarrow \int_a^b f(x)\,dx \geq 0$

- Dacă $m \leq f(x) \leq M, \ \forall x \in [a,b]$

  atunci $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

  $\hookrightarrow$ proprietatea de medie.

Exerciții rezolvate

① $\boxed{I = \int_0^1 (2x^2 - x - 3)\,dx}$

$$I = \int_0^1 2x^2\,dx - \int_0^1 x\,dx - \int_0^1 3\,dx = 2\int_0^1 x^2\,dx -$$

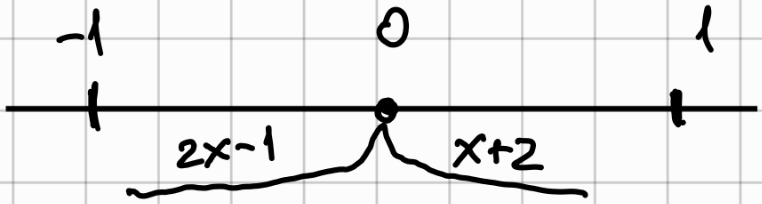
$$- \int_0^1 x\,dx - 3\int_0^1 dx = 2 \cdot \frac{x^3}{3}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 - 3x\Big|_0^1 =$$

$$= 2 \cdot \left(\frac{1^3}{3} - \frac{0^3}{3}\right) - \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - 3(1-0) =$$

$$= {}^{2)}\frac{2}{3} - {}^{3)}\frac{1}{2} - {}^{6)}3 = \frac{4-3-18}{6} = \frac{-17}{6}$$

② $f : [-1,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 2x-1, & x \leq 0 \\ x+2, & x > 0 \end{cases}$

Calculați $\int_{-1}^1 f(x)\,dx$



$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx =$$

$$= \int_{-1}^{0} (2x-1)\,dx + \int_{0}^{1} (x+2)\,dx =$$

$$= 2\int_{-1}^{0} x\,dx - \int_{-1}^{0} dx + \int_{0}^{1} x\,dx + 2\int_{0}^{1} dx =$$

$$= \cancel{2}\cdot \frac{x^2}{\cancel{2}}\Big|_{-1}^{0} - x\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1} + 2x\Big|_{0}^{1} =$$

$$= \left[0^2-(-1)^2\right] - \left[0-(-1)\right] + \left(\frac{1^2}{2} - \frac{0^2}{2}\right) + 2(1-0)$$

$$= -\cancel{1} - \cancel{1} + \frac{1}{2} + \cancel{2} = \frac{1}{2}$$

3 Comparați numerele

$a = \int_{0}^{1} (x^2-1)\,dx$ și $b = \int_{0}^{1} (x^2+x)\,dx$

Folosim proprietatea de monotonie.

Fie $f, g : [0,1] \to \mathbb{R}$, $f(x) = x^2-1$

$g(x) = x^2+x$.

$$g(x) - f(x) = x^2+x-(x^2-1) = \cancel{x^2}+x-\cancel{x^2}+1 = x+1$$

$$g(x) - f(x) = x+1$$

$$x \in [0,1] \Rightarrow 0 \le x \le 1 \quad |+1$$

$$1 \le x+1 \le 2 \quad \Rightarrow x+1 \ge 0$$

Cum $x+1 \geq 0 \Rightarrow g(x) - f(x) \geq 0 \Rightarrow$

$$\Rightarrow g(x) \geq f(x) \Rightarrow \int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx \Rightarrow$$

$$\Rightarrow b \geq a.$$