Derivata funcției inverse - aplicații

1. Fie funcția $f:\mathbb{R}\to(1,+\infty)$, $f(x)=5^x+3^x+1$.
a) Arătați că f este inversabilă;
b) Calculați $(f^{-1})'(3)$.

Rezolvare:

Dacă $f'(x)>0 \Rightarrow f$ strict crescătoare
$f'(x)\geq 0 \Rightarrow f$ crescătoare
Dacă $f'(x)<0 \Rightarrow f$ strict descrescătoare
$f'(x)\leq 0 \Rightarrow f$ descrescătoare

a)
$$f'(x)=(5^x+3^x+1)'=(5^x)'+(3^x)'+1'=\underbrace{5^x}_{>0}\underbrace{\ln 5}_{>0}+\underbrace{3^x}_{>0}\underbrace{\ln 3}_{>0}>0 \Rightarrow$$

$\Rightarrow f'(x)>0 \Rightarrow f$ s.c. pe $\mathbb{R} \Rightarrow f$ inj (1)

Pt studiul surjectivității, întocmim tabelul de valori.

x	$-\infty$					$+\infty$
$f'(x)$	+	+	+	+	+	
$f(x)$	1		$\nearrow$			$+\infty$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}(5^x+3^x+1)=\underbrace{5^{-\infty}}_{0}+\underbrace{3^{-\infty}}_{0}+1=1.$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}(5^x+3^x+1)=5^{\infty}+3^{\infty}+1=\infty$$

f cont pe $\mathbb{R}$ (sumă de fct elementare)

$\Rightarrow \operatorname{Im} f=(1;+\infty)$
$\Downarrow$
f - surj (2)

Din (1) și (2) $\Rightarrow f$ - bijectivă $\Rightarrow f$ inversabilă

b) $\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \quad y_0 = f(x_0)}$ $\quad (f^{-1})'(3) = ?$

$\left.\begin{array}{l} y_0 = 3 \\ y_0 = f(x_0) \end{array}\right| \Rightarrow f(x_0) = 3 \Rightarrow 5^{x_0} + 3^{x_0} + 1 = 3$

$5^{x_0} + 3^{x_0} = 2$

Intuim soluția:

$x_0 = 0 \Rightarrow 5^0 + 3^0 = 2$ (A) $\Rightarrow$

$\Rightarrow \boxed{x_0 = 0}$ soluție.

Dem. că $x_0 = 0$ este soluție unică.

Fie $h: \mathbb{R} \to \mathbb{R}$, $h(x) = 5^x + 3^x$ $\quad \Rightarrow h(x) = 2$

Pp $\left.\begin{array}{l} x < 0 \\ h \text{ s.c} \end{array}\right| \Rightarrow h(x) < h(0) \Rightarrow 5^x + 3^x < 2$ (F) pt că $h(x) = 2 \Rightarrow \nexists x_0 < 0$.

Pp $\left.\begin{array}{l} x > 0 \\ h \text{ s.c} \end{array}\right| \Rightarrow h(x) > h(0) \Rightarrow 5^x + 3^x > 2$ (F) pt că $h(x) = 2 \Rightarrow \nexists x_0 > 0$

$\left.\begin{array}{l} x_0 = 0 \\ f'(x) = 5^x \ln 5 + 3^x \ln 3 \end{array}\right| \Rightarrow f'(0) = 5^0 \ln 5 + 3^0 \ln 3$

$f'(0) = \ln 5 + \ln 3$

$f'(0) = \ln 15$

$(f^{-1})'(3) = \frac{1}{f'(0)} = \frac{1}{\ln 15}$

2. Fie $f: \left[\frac{3}{2}, +\infty\right) \to \left[-\frac{5}{4}, +\infty\right)$, $f(x) = x^2 - 3x + 1$.

a) Arătați că f este inversabilă;

b) Calculați $(f^{-1})'(5)$.

Rezolvare:

a) $f'(x) = (x^2-3x+1)' = 2x-3$

$x \in \left[\frac{3}{2}, +\infty\right) \Rightarrow x \geq \frac{3}{2} \Rightarrow 2x \geq 3 \Rightarrow 2x-3 \geq 0 \Rightarrow f'(x) \geq 0 \Rightarrow$

$\Rightarrow f$ crescătoare pe $\left[\frac{3}{2}, +\infty\right) \Rightarrow f$ - inj (1)

x	$\left[\frac{3}{2}\right.$					$+\infty$
$f'(x)$	+	+	+	+	+	+
$f(x)$	$\left[-\frac{5}{4}\right.$	$\nearrow$				$+\infty$

$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2} + 1 = \frac{9}{4} - \frac{9}{2} + 1 = -\frac{5}{4}$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (x^2-3x+1) = \lim\limits_{x\to\infty} x^2\left(1 - \underbrace{\frac{3}{x}}_{\to 0} + \underbrace{\frac{1}{x^2}}_{\to 0}\right) = \infty^2 = \infty$ $\Rightarrow$

f - cont pe $\left[\frac{3}{2}; +\infty\right)$ (compunere de fct elementare)

$\Rightarrow \operatorname{Im} f = \left[-\frac{5}{4}, +\infty\right) \Rightarrow f$ - surj (2)

Din (1) și (2) $\Rightarrow f$ bij $\Rightarrow f$ inversabilă

b) $(f^{-1})'(5) = ?$

$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$ $\Rightarrow y_0 = 5 \Rightarrow f(x_0) = 5$

$x_0^2 - 3x_0 + 1 = 5$

$x_0^2 - 3x_0 - 4 = 0$

$\Delta = 25$, $x_0' = \frac{3-5}{2} = -1$, $x_0'' = \frac{3+5}{2} = 4$

$x_0' = -1$, $f'(x) = 2x-3$ $\Rightarrow f'(-1) = -2-3 = -5 \Rightarrow (f^{-1})'(5) = -\frac{1}{5}$

$x_0'' = 4$, $f'(x) = 2x-3$ $\Rightarrow f'(4) = 8-3 = 5 \Rightarrow (f^{-1})'(5) = \frac{1}{5}$

3. Calculați derivatele:

a) $\left[\arcsin(2x-5)\right]' = \dfrac{(2x-5)'}{\sqrt{1-(2x-5)^2}} = \dfrac{2}{\sqrt{-4x^2+20x-24}}$

$\boxed{(\arcsin \mu)' = \dfrac{\mu'}{\sqrt{1-\mu^2}}}$

$= \dfrac{\cancel{2}}{\cancel{2}\sqrt{-x^2+5x-6}} = \dfrac{1}{\sqrt{-x^2+5x-6}}$

b) $\left(\arcsin^2 \dfrac{1}{x+1}\right)' = 2\arcsin\dfrac{1}{x+1}\cdot\left(\arcsin\dfrac{1}{x+1}\right)' =$

$= 2\arcsin\dfrac{1}{x+1}\cdot\dfrac{\left(\frac{1}{x+1}\right)'}{\sqrt{1-\left(\frac{1}{x+1}\right)^2}} = 2\arcsin\dfrac{1}{x+1}\cdot\dfrac{\frac{\cancel{\overset{0}{1'}\cdot(x+1)}-1\cdot(x+1)'}{(x+1)^2}}{\sqrt{1-\frac{1}{(x+1)^2}}}$

$= 2\arcsin\dfrac{1}{x+1}\cdot\dfrac{\frac{-1}{(x+1)^2}}{\sqrt{\frac{x^2+2x}{(x+1)^2}}} = 2\arcsin\dfrac{1}{x+1}\cdot\dfrac{-1}{(x+1)^{\cancel{2}}}\cdot\dfrac{\cancel{x+1}}{\sqrt{x^2+2x}} =$

$= \dfrac{-2\arcsin\frac{1}{x+1}}{(x+1)\sqrt{x(x+2)}}$

$\boxed{(\arccos \mu)' = \dfrac{-\mu'}{\sqrt{1-\mu^2}}}$

c) $\left[\arccos\left(e^{x+2}\right)\right]' = \dfrac{-\left(e^{x+2}\right)'}{\sqrt{1-e^{2(x+2)}}} = \dfrac{-e^{x+2}\cdot(x+2)'}{\sqrt{1-e^{2x+4}}} =$

$= \dfrac{-e^{x+2}}{\sqrt{1-e^{2x+4}}}$

d) $\left[\arccos^4(x^2-1)\right]' = 4\arccos^3(x^2-1)\cdot\left[\arccos(x^2-1)\right]' =$

$= 4\arccos^3(x^2-1)\cdot\dfrac{-(x^2-1)'}{\sqrt{1-(x^2-1)^2}} = \dfrac{-8x\arccos^3(x^2-1)}{\sqrt{-x^4+2x^2}} =$

$$= \frac{-8x \arccos^3(x^2-1)}{|x|\sqrt{2-x^2}}$$

$$\boxed{(\text{arctg}\, u)' = \frac{u'}{1+u^2}}$$

e) $$\left(\text{arctg}\,\frac{1-x^2}{1+x^2}\right)' = \frac{\left(\frac{1-x^2}{1+x^2}\right)'}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} = \frac{\frac{-2x(1+x^2)-(1-x^2)\cdot 2x}{\cancel{(1+x^2)^2}}}{\frac{(1+x^2)^2+(1-x^2)^2}{\cancel{(1+x^2)^2}}}$$

$$= \frac{-2x\cancel{-2x^3}-2x\cancel{+2x^3}}{1\cancel{+2x^2}+x^4+1\cancel{-2x^2}+x^4} = \frac{-4}{2+2x^4} = \frac{-2}{1+x^4}$$

$$\boxed{(\text{arcctg}\, u)' = \frac{-u'}{1+u^2}}$$

f) $$\left(\text{arcctg}\, e^{-2x}\right)' = \frac{-(e^{-2x})'}{1+(e^{-2x})^2} = \frac{-e^{-2x}\cdot(-2x)'}{1+e^{-4x}} =$$

$$= \frac{2e^{-2x}}{1+\frac{1}{e^{4x}}} = \frac{2\cdot e^{-2x}}{\frac{e^{4x}+1}{e^{4x}}} = \frac{2e^{-6x}}{1+e^{4x}}$$

4. Calculați derivatele funcțiilor $f: D_f \to \mathbb{R}$, precizând D_f și D_f' pentru fiecare:

a) $$\boxed{f(x) = \frac{x(1+x)}{\sqrt{1-x}}}$$

C.E: $1-x>0 \Rightarrow x<1 \Rightarrow D_f = (-\infty; 1)$

$$f'(x) = \frac{[x(1+x)]'\cdot\sqrt{1-x} - x(1+x)\cdot\sqrt{1-x}\,'}{1-x} = \frac{(1+2x)\sqrt{1-x} - x(1+x)\cdot\frac{-1}{2\sqrt{1-x}}}{1-x} =$$

$2\sqrt{1-x}$

$$= \frac{\dfrac{(2+4x)(1-x) + x(x+1)}{2\sqrt{1-x}}}{1-x} = \frac{2-2x+4x-4x^2+x^2+x}{2(1-x)\sqrt{1-x}} = \frac{-3x^2+3x+2}{2(1-x)\sqrt{1-x}}$$

$C.E: \ 1-x>0 \Rightarrow x<1 \Rightarrow D_{f'} = (-\infty; 1)$

b) $\boxed{f(x) = \sqrt{x\sqrt{x}}}$

$$f(x) = \sqrt{x\sqrt{x}} = \sqrt{\sqrt{x \cdot x^2}} = \sqrt[4]{x^3}$$

$C.E: \ x^3 \geq 0 \Rightarrow x \geq 0 \Rightarrow D_f = [0; +\infty)$

$$f'(x) = \left(\sqrt[4]{x^3}\right)' = \frac{1}{4\sqrt[4]{(x^3)^3}} \cdot (x^3)' = \frac{3x^2}{4\sqrt[4]{x^9}} = \frac{3x^2}{4\sqrt[4]{x^8 \cdot x}} = \frac{3\cancel{x^2}}{4\cancel{x^2}\sqrt[4]{x}} =$$

$$= \frac{3}{4\sqrt[4]{x}}$$

$C.E: \ x>0 \Rightarrow D_{f'} = (0, +\infty)$

c) $\boxed{f(x) = \dfrac{\sqrt{x^3-x}}{x+2}}$

$$C.E: \begin{cases} x^3 - x \geq 0 \\ x+2 \neq 0. \end{cases} \Rightarrow \begin{cases} x(x^2-1) \geq 0 \\ x \neq -2 \end{cases} \Rightarrow \begin{cases} x(x-1)(x+1) \geq 0 \\ x \neq -2 \end{cases}$$

x	$-\infty$		-2		-1		0		1		$+\infty$
x		−	−	−	−	−	0	+	+	+	+
$x-1$		−	−	−	−	−	−	−	0	+	+
$x+1$		−	−	−	0	+	+	+	+	+	+
x^3-x		−			0	+	0	−	0	+	

$$D_f = [-1, 0] \cup [1, +\infty)$$

$$f'(x) = \left(\frac{\sqrt{x^3-x}}{x+2}\right)' = \frac{\left(\sqrt{x^3-x}\right)' \cdot (x+2) - \sqrt{x^3-x} \cdot (x+2)'}{(x+2)^2} =$$

$$= \frac{\frac{1}{2\sqrt{x^3-x}} \cdot (x^3-x)' \cdot (x+2) - \frac{2\sqrt{x^3-x}}{\sqrt{x^3-x}}}{(x+2)^2} = \frac{(3x^2-1)(x+2) - 2(x^3-x)}{2(x+2)^2\sqrt{x^3-x}} =$$

$$= \frac{3x^3+6x^2-x-2-2x^3+2x}{2(x+2)^2\sqrt{x^3-x}} = \frac{x^3+6x^2+x-2}{2(x+2)^2\sqrt{x^3-x}}$$

C.E. $\begin{cases} x^3-x>0 \\ x+2\neq 0. \end{cases} \Rightarrow D_f = (-1,0)\cup(1,+\infty)$

d) $\boxed{f(x) = \left(\frac{1}{x}\right)^{x+1}}$

$$(f^g)' = g f^{g-1} \cdot f' + f^g \cdot \ln f \cdot f'$$

C.E: $x\neq 0 \Rightarrow D_f = \mathbb{R}\setminus\{0\}$

$$f'(x) = \left[\left(\frac{1}{x}\right)^{x+1}\right]' = (x+1)\cdot\left(\frac{1}{x}\right)^{x}\cdot\left(\frac{1}{x}\right)' + \left(\frac{1}{x}\right)^{x+1}\ln\frac{1}{x}\cdot\left(\frac{1}{x}\right)' =$$

$$= \left(\frac{1}{x}\right)'\cdot\left[(x+1)\cdot\left(\frac{1}{x}\right)^{x} + \left(\frac{1}{x}\right)^{x}\cdot\frac{1}{x}\ln\frac{1}{x}\right] = \frac{\cancel{1'x}-1\cdot x'}{x^2}\cdot\left(\frac{1}{x}\right)^{x}\cdot\left(x+1+\frac{1}{x}\ln\frac{1}{x}\right) =$$

$$= -\frac{1}{x^2}\cdot x^{-x}\cdot\left(x+1+\frac{1}{x}\ln\frac{1}{x}\right) = -x^{-x-2}\left(x+1+\frac{1}{x}\ln\frac{1}{x}\right)$$

C.E: $\begin{cases} x\neq 0 \\ \frac{1}{x}>0 \Rightarrow x>0 \end{cases} \Rightarrow D_{f'} = (0,+\infty)$

e) $\boxed{f(x) = \arcsin\sqrt{1-x^2}}$

C.E: $1-x^2\geq 0 \Rightarrow x^2\leq 1.$

x	$-\infty$		-1		1		$+\infty$
x^2-1		$+$	0	$-$	0	$+$	

$\Rightarrow x\in[-1,1]$

Funcţia arcsin$t$ există dacă $t\in[-1,1]$

$\sqrt{1-x^2} = t \Rightarrow t^2 = 1-x^2 \Rightarrow t^2\geq 0 \Rightarrow t\geq 0 \quad \Big| \Rightarrow$

$\Rightarrow t \in [0,1]$.
Dar $[0,1] \subset [-1,1]$ $\Big| \Rightarrow \boxed{D_f = [-1,1]}$

$$f'(x) = \left(\arcsin\sqrt{1-x^2}\right)' = \frac{\sqrt{1-x^2}\,'}{\sqrt{1-(1-x^2)}} =$$

$$= \frac{\frac{1}{2\sqrt{1-x^2}} \cdot (1-x^2)'}{\sqrt{x^2}} = \frac{-x}{|x| \cdot \sqrt{1-x^2}}$$

C.E.: $1-x^2 > 0$
$|x| \neq 0 \Rightarrow x \neq 0$

x	$-\infty$		-1		1		$+\infty$
$1-x^2$	$-$	$-$ $-$	0	$+$	0	$-$ $-$ $-$	$-$

$\Rightarrow x \in (-1,1)$

$\left.\begin{array}{l} x \neq 0 \\ x \in (-1,1) \end{array}\right| \Rightarrow x \in (-1,0) \cup (0,1) \Rightarrow \boxed{D_{f'} = (-1,0) \cup (0,1)}$