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Baze. Matricea de trecere de la o bază la alta

O mulțime $B = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ este sistem de generatori pentru spațiul vectorial V dacă orice vector din V se poate scrie ca o combinație liniară a vect din B

$$\left.\begin{array}{l} \forall \bar{v} \in V \\ \exists\, \alpha_1, \alpha_2, \dots, \alpha_n \in K \end{array}\right| \Rightarrow \bar{v} = \alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \dots + \alpha_n \bar{e}_n$$

Spunem că $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ formează o bază pt K spațiul vectorial V dacă și numai dacă:

1. constituie un sistem de generatori
2. sunt l.i.

Ex 1

Fie $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \subset \mathbb{R}^3$, $\bar{e}_1 = (1,2,-1)$, $\bar{e}_2 = (2,-1,2)$, $\bar{e}_3 = (-1,2,2)$

Să se arate că B este o bază a sp. vectorial $\mathbb{R}^3$

1. B sistem de generatori

$$\bar{v} = \alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \alpha_3 \bar{e}_3, \quad \bar{v} = (x,y,z) \in \mathbb{R}^3$$

$$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

$$(x,y,z) = \alpha_1 \cdot (2,2,-1) + \alpha_2 (2,-1,2) + \alpha_3 (-1,2,2)$$

$$(x,y,z) = (2\alpha_1, 2\alpha_1, -\alpha_1) + (2\alpha_2, -\alpha_2, 2\alpha_2) + (-\alpha_3, 2\alpha_3, 2\alpha_3)$$

$$\begin{cases} 2\alpha_1 + 2\alpha_2 - \alpha_3 = x \\ 2\alpha_1 - \alpha_2 + 2\alpha_3 = y \\ -\alpha_1 + 2\alpha_2 + 2\alpha_3 = z \end{cases} \qquad A = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{vmatrix} = -4-4-4-(-1+8+8) = -12-15 = -27 \neq 0$$

$\Rightarrow$ S.C.D. sol unică (Cramer)

$$\alpha_1 = \frac{\Delta\alpha_1}{\det A}, \quad \Delta\alpha_1 = \begin{vmatrix} x & 2 & -1 \\ y & -1 & 2 \\ z & 2 & 2 \end{vmatrix} = -2x-2y+4z-(z+4x+4y) = -6x-6y+3z$$

$$\alpha_1 = \frac{\cancel{3}^{1}(-2x-2y+z)}{\cancel{-27}\ 9} = \frac{-2x-2y+z}{-9} = \frac{2x+2y-z}{9}$$

$$\alpha_2 = \frac{\Delta\alpha_2}{\det A}, \quad \Delta\alpha_2 = \begin{vmatrix} 2 & x & -1 \\ 2 & y & 2 \\ -1 & z & 2 \end{vmatrix} = 4y-2z-2x-(y+4x+4z) = -6x+3y-6z$$

$$\alpha_2 = \frac{\cancel{3}^{1}(-2x+y-2z)}{\cancel{-27}\ 9} = \frac{-2x+y-2z}{-9} = \frac{2x-y+2z}{9}$$

$$\alpha_3 = \frac{\Delta\alpha_3}{\det A}, \quad \Delta\alpha_3 = \begin{vmatrix} 2 & 2 & x \\ 2 & -1 & y \\ -1 & 2 & z \end{vmatrix} = -2z+4x-2y-(x+4y+4z) = 3x-6y-6z$$

$$\alpha_3 = \frac{\cancel{3}^{1}(x-2y-2z)}{\cancel{-27}\ 9} = \frac{x-2y-2z}{-9} = \frac{-x+2y+2z}{9}$$

$$\bar{v} = \left(\frac{2x+2y-z}{9} e_1 + \frac{2x-y+2z}{9} e_2 + \frac{-x+2y+2z}{9} e_3\right)$$

$\Rightarrow$ B - sistem de generatori

2. Studiem liniar independența vectorilor

$$\alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \alpha_3 \bar{e}_3 = \bar{0}$$

$e_1 = (2,2,-1)$
$e_2 = (2,-1,2)$
$e_3 = (-1,2,2)$

$$\Rightarrow \begin{cases} 2\alpha_1 + 2\alpha_2 - \alpha_3 = 0 \\ 2\alpha_1 - \alpha_2 + 2\alpha_3 = 0 \\ -\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 \end{cases}, \quad A = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}, \ \det A = -27 \neq 0$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ (sist omogen)

$\Rightarrow \bar{e}_1, \bar{e}_2, \bar{e}_3$ l.i.

Din (1) și (2) $\Rightarrow$ B - bază a lui $\mathbb{R}^3$

Exerciții propuse:

① $B = \{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$

B – bază a lui $\mathbb{R}^3$?

② $B = \{(1,1,0), (1,0,1), (0,1,1)\}$ este bază a lui $\mathbb{R}^3$?

③ $B = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$ $\quad e_1 = (2,-1,0)$

$e_2 = (1,1,-1)$

$e_3 = (-1,2,1)$

B – bază a lui $\mathbb{R}^3$?

Exp 2

> Fie vectorul $\bar{v} = (1,1,1) \in \mathbb{R}^3$
> și baza $B = \{e_1 = (1,0,-1),\ e_2 = (-1,1,1),\ e_3 = (2,1,0)\}$
> Determinați coordonatele lui $\bar{v}$ în baza B

Căutăm $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ a.î. $\bar{v} = \alpha_1 \bar{e_1} + \alpha_2 \bar{e_2} + \alpha_3 \bar{e_3}$

$(1,1,1) = \alpha_1(1,0,-1) + \alpha_2(-1,1,1) + \alpha_3(2,1,0)$

$(1,1,1) = (\alpha_1 - \alpha_2 + 2\alpha_3,\ \alpha_2 + \alpha_3,\ -\alpha_1 + \alpha_2)$

$$\begin{cases} \alpha_1 - \alpha_2 + 2\alpha_3 = 1 \\ \alpha_2 + \alpha_3 = 1 \\ -\alpha_1 + \alpha_2 = 1 \end{cases}, \quad A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 0 + 0 + 1 - (-2 - 0 + 1) = 1 + 1 = 2 \neq 0$$

$\Rightarrow$ S.C.D. (Cramer)

$$\alpha_1 = \frac{\Delta\alpha_1}{\det A}, \quad \Delta\alpha_1 = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 + 2 - 1 - (2 + 1 - 0) = -2$$

$$\alpha_1 = \frac{-2}{2} = -1$$

$$\alpha_2 = \frac{\Delta\alpha_2}{\det A}, \quad \Delta\alpha_2 = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 0 + 0 - 1 - (-2 + 1 + 0) = 0$$

$$\alpha_2 = \frac{0}{2} = 0$$

$$\alpha_3 = \frac{\Delta\alpha_3}{\det A}, \quad \Delta\alpha_3 = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1+0+1-(\cancel{-1}+\cancel{1}-0) = 2$$

$$\alpha_3 = \frac{2}{2} = 1$$

$$\bar{v} = -\bar{e}_1 + 0\bar{e}_2 + \bar{e}_3$$

Ex2

Fie polinomul $f = X^5 - X^4 + X^3 - X^2 + X - 1 \in \mathbb{R}_5[X]$ și baza $B = \{1, 1+X, 1+X^2, 1-X^3, 1+X^4, 1+X^5\}$

Determinați coordonatele lui f în baza B.

$\mathbb{R}_5[X]$ - mult. polinoamelor de grad cel mult 5.

$$\bar{v} = \alpha_1\bar{e}_1 + \alpha_2\bar{e}_2 + \alpha_3\bar{e}_3 + \alpha_4\bar{e}_4 + \alpha_5\bar{e}_5 + \alpha_6\bar{e}_6$$

$$\bar{v} = \alpha_1\cdot 1 + \alpha_2(1+X) + \alpha_3(1+X^2) + \alpha_4(1-X^3) + \alpha_5(1+X^4) + \alpha_6(1+X^5)$$

$$\Rightarrow \bar{v} = \alpha_1 + \alpha_2 + \alpha_2 X + \alpha_3 + \alpha_3 X^2 + \alpha_4 - \alpha_4 X^3 + \alpha_5 + \alpha_5 X^4 + \alpha_6 + \alpha_6 X^5$$

$$\bar{v} = -1 + X - X^2 + X^3 - X^4 + X^5 \quad \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = -1 \\ \alpha_2 = 1 \\ \alpha_3 = -1 \\ -\alpha_4 = 1 \\ \alpha_5 = -1 \\ \alpha_6 = 1 \end{cases}$$

$$\Rightarrow \alpha_1 + \cancel{1} \cancel{-1} - 1 \cancel{-1} + \cancel{1} = -1 \Rightarrow \alpha_1 = 0$$

$$f = 0\bar{e}_1 + \bar{e}_2 - \bar{e}_3 - \bar{e}_4 - \bar{e}_5 + e_6$$

Exerciții propuse:

1. $f = X^3 - 2X^2 + X - 1 \in \mathbb{R}_3[X]$, $B = \{1, 1+X, 1+X^2, 1-X^3\}$

   Determinați coord. lui f în baza B.

2. $f = 2X^4 - X^3 + X^2 - 2X + 1 \in \mathbb{R}_4[X]$, $B = \{1, 2-X, 1+X^2, -3+X^3, -1-X^4\}$

   Determinați coord. lui f în baza B.

Ex 3

Îm $\mathbb{R}^3$ se consideră

$B' = \{ e_1' = (1,1,0),\ e_2' = (1,0,1),\ e_3' = (1,0,-1) \}$

$B'' = \{ e_1'' = (1,0,0),\ e_2'' = (1,1,0),\ e_3'' = (1,1,1) \}$

a) Arătați că B', B'' - baze în $\mathbb{R}^3$

b) Să se determine matricea de trecere de la baza B' la baza B''

c) Determinați coordonatele vect. $\bar{v} = (2,-1,1)$ în bazele B' și B''. Vectorul $\bar{v}$ este dat prin coordonatele sale în baza canonică a lui $\mathbb{R}^3$.

a) $\underline{B' - \text{bază}}$ $\iff$ 1. B' - sist de generatori
2. B' - l.i.

1. $\forall \bar{v} \in \mathbb{R}^3$, $\bar{v} = \alpha_1 \bar{e}_1' + \alpha_2 \bar{e}_2' + \alpha_3 \bar{e}_3'$

$\bar{v} = (x', y', z')$, etc $\alpha_1, \alpha_2, \alpha_3 = \dots$

2. $\alpha_1 \bar{e}_1' + \alpha_2 \bar{e}_2' + \alpha_3 \bar{e}_3' = 0 \Rightarrow \dots\ \alpha_1 = \alpha_2 = \alpha_3 = 0$

Asemănător $\underline{B'' - \text{bază}}$ //

b) Fie C - matricea de trecere de la B' la B''

Obs: Trebuie ca fiecare vector din B' să fie exprimat ca o combinație liniară a vect din B''.

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$e_1' = (1,1,0)$	$e_1'' = (1,0,0)$
$e_2' = (1,0,1)$	$e_2'' = (1,1,0)$
$e_3' = (1,0,-1)$	$e_3'' = (1,1,1)$

$e_1' = c_{11} e_1'' + c_{21} e_2'' + c_{31} e_3''$

$e_2' = c_{12} e_1'' + c_{22} e_2'' + c_{23} e_3''$

$e_3' = c_{13} e_1'' + c_{23} e_2'' + c_{33} e_3''$

$e_1' = c_{11} e_1'' + c_{21} e_2'' + c_{31} e_3''$

$(1,1,0) = c_{11}(1,0,0) + c_{21}(1,1,0) + c_{31}(1,1,1)$

$(1,1,0) = (c_{11} + c_{21} + c_{31},\ c_{21} + c_{31},\ c_{31}) \Rightarrow \begin{cases} c_{11} + c_{21} + c_{31} = 1 \\ c_{21} + c_{31} = 1 \\ c_{31} = 0 \end{cases} \Rightarrow$

$\Rightarrow c_{21} + 0 = 1 \Rightarrow c_{21} = 1$

$c_{11} + 1 + 0 = 1 \Rightarrow c_{11} = 0$

$e_2' = c_{12} e_1'' + c_{22} e_2'' + c_{23} e_3''$

$(1,0,1) = c_{12}(1,0,0) + c_{22}(1,1,0) + c_{23}(1,1,1)$

$(1,0,1) = (c_{12} + c_{22} + c_{23},\ c_{22} + c_{23},\ c_{23}) \Rightarrow \begin{cases} c_{12} + c_{22} + c_{23} = 1 \\ c_{22} + c_{23} = 0 \\ c_{23} = 1 \end{cases}$

$\Rightarrow c_{22} + 1 = 0 \Rightarrow c_{22} = -1$

$c_{12} + (-1) + 1 = 0 \Rightarrow c_{12} = 0$

$e_3' = c_{13} e_1'' + c_{23} e_2'' + c_{33} e_3''$

$(1,0,-1) = c_{13}(1,0,0) + c_{23}(1,1,0) + c_{33}(1,1,1)$

$(1,0,-1) = (c_{13} + c_{23} + c_{33},\ c_{23} + c_{33},\ c_{33}) \Rightarrow \begin{cases} c_{13} + c_{23} + c_{33} = 1 \\ c_{23} + c_{33} = 0 \\ c_{33} = -1 \end{cases}$

$\Rightarrow c_{23} - 1 = 0 \Rightarrow c_{23} = 1$

$c_{13} + 1 - 1 = 1 \Rightarrow c_{13} = 1$

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Pentru verificare: $C = B' \cdot (B'')^{-1} \;\big|\cdot B'' \Rightarrow C \cdot B'' = B'$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}}_{C} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{B''} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}}_{B'} \quad \text{adev.}$$

c) <u>coord vect $\bar{v} = (2,-1,1)$ în baza B'</u>

$e_1' = (1,1,0)$
$e_2' = (1,0,1)$
$e_3' = (1,0,-1)$

$\bar{v} = \alpha_1 \bar{e}_1' + \alpha_2 \bar{e}_2' + \alpha_3 \bar{e}_3'$

$(2,-1,1) = \alpha_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(1,0,-1)$

$(2,-1,1) = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1, \alpha_2 - \alpha_3) \Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 2 \\ \alpha_1 = -1 \\ \alpha_2 - \alpha_3 = 1 \end{cases}$

$\Rightarrow \begin{cases} \alpha_2 + \alpha_3 = 3 \\ \alpha_2 - \alpha_3 = 1 \end{cases}$ (+) $\Rightarrow 2 + \alpha_3 = 3 \Rightarrow \alpha_3 = 1$

$2\alpha_2 = 4 \Rightarrow \alpha_2 = 2$

$\bar{v}_{B'} = (-1,2,1)$

Analog se determină $\bar{v}_{B''}$

<u>Exerciții propuse</u>

① În $\mathbb{R}^3$ se consideră

$B' = \{ e_1' = (-1,0,1),\ e_2' = (0,-1,1),\ e_3' = (-1,-1,0) \}$

$B'' = \{ e_1'' = (0,1,1),\ e_2'' = (-1,0,0),\ e_3'' = (1,1,1) \}$

a) Arătați că B', B'' - baze în $\mathbb{R}^3$

b) Să se determine matricea de trecere de la baza B' la baza B''

c) Determinați coordonatele vect. $\bar{v} = (-1,1,2)$ în bazele B' și B''.

② $B = \{ e_1 = (1,-1,0),\ e_2 = (1,0,-1),\ e_3 = (0,1,1) \}$

$B' = \{ e_1' = (1,0,0),\ e_2' = (-1,-1,0),\ e_3' = (0,1,1) \}$

a) B, B' - baze

b) Să se determine matricea de trecere de la baza B' la baza B.

c) Determinați coordonatele vect. $\bar{v} = (-3,1,-1)$ în bazele B și B'.

(3) Precizați care dintre următoarele sisteme de vectori formează baze în spațiile vectoriale date și care este dimensiunea fiecăruia:

a) $S_1 = \{\bar{u}_1, \bar{u}_2\} \subset \mathbb{R}^2$, $\bar{u}_1 = (-1,1)$
$\bar{u}_2 = (2,-1)$

b) $S_2 = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\} \subset \mathbb{R}^3$, $\bar{v}_1 = (-1,1,1)$
$\bar{v}_2 = (0,-1,1)$
$\bar{v}_3 = (2,1,-1)$

c) $S_3 = \{1, 1+X, (1+X)^2, 1-X^3\} \subset \mathbb{R}_3[X]$

$$\boxed{\dim V = \begin{cases} 0, & V = \{\bar{0}\} \\ n, & V \neq \{\bar{0}\} \text{ și admite o bază de } n \text{ vectori} \\ \infty, & V \text{ - nu este de tip finit} \end{cases}}$$

Exp: 1) $V = \{\bar{0}\}$ - spațiul V conține doar vectorul nul
$B = \emptyset$ (baza este mulțimea vidă)
$\dim V = 0$.

2) $\mathbb{R}^4$, $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$
dacă S - bază $\Rightarrow \dim S = 4$

3) $\mathbb{R}_n[X]$, $S = \{1, X, X^2, X^3, \dots\}$
Baza conține n vectori $\Rightarrow \dim S = \infty$

4) $\mathbb{R}_3[X]$, $S = \{1, X, X^2, X^3\}$

Baza conține 4 vectori $\Rightarrow \dim S = \dim \mathbb{R}_3[X] = 4$

Ex4 d) $S_4 = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A_1}, \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{A_2}, \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A_3}, \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{A_4} \right\} \subset \mathcal{M}_2(\mathbb{R})$

Rezolvare:

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = O_2$$

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \alpha_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_3 & \alpha_3 \\ 0 & \alpha_3 \end{pmatrix} + \begin{pmatrix} \alpha_4 & \alpha_4 \\ \alpha_4 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 \\ \alpha_4 & \alpha_3 + \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 & \Rightarrow \alpha_1 = 0 \\ \alpha_2 + \alpha_3 + \alpha_4 = 0 & \Rightarrow \alpha_2 = 0 \\ \alpha_4 = 0 & \\ \alpha_3 + \alpha_4 = 0. & \Rightarrow \alpha_3 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

$\Downarrow$

S_4 - bază

$\dim S_4 = 4$

$\mathcal{M}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / a, b, c, d \in \mathbb{R} \right\} \Rightarrow \dim \mathcal{M}_2(\mathbb{R}) = 4$ $\Bigg| \Rightarrow$

$\Rightarrow \dim S = \dim \mathcal{M}_2(\mathbb{R}) = 4$

Ex.5

Se consideră baza $B = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$, $\bar{v}_1 = (1,2,1)$

$\bar{v}_2 = (2,3,3)$

$\bar{v}_3 = (3,7,1)$

Determinați coordonatele vect $\bar{x} = (3,-1,2)$ în baza B

Metoda 1: $\bar{x} = \alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \alpha_3 \bar{v}_3$

$(3,-1,2) = (\alpha_1, 2\alpha_1, \alpha_1) + (2\alpha_2, 3\alpha_2, 3\alpha_2) + (3\alpha_3, 7\alpha_3, \alpha_3)$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + 3\alpha_3 = 3 \\ 2\alpha_1 + 3\alpha_2 + 7\alpha_3 = -1 \\ \alpha_1 + 3\alpha_2 + \alpha_3 = 2 \end{cases}, \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ 1 & 3 & 1 \end{pmatrix}, \ \det A = 1 \neq 0$$

$\Downarrow$

S.C.D, sol unică (Cramer)

$$\alpha_1 = \frac{\Delta\alpha_1}{\det A}, \quad \Delta\alpha_1 = \begin{vmatrix} 3 & 2 & 3 \\ -1 & 3 & 7 \\ 2 & 3 & 1 \end{vmatrix} = -51, \quad \alpha_1 = -51$$

$$\alpha_2 = \frac{\Delta\alpha_2}{\det A}, \quad \Delta\alpha_2 = \begin{vmatrix} 1 & 3 & 3 \\ 2 & -1 & 7 \\ 1 & 2 & 1 \end{vmatrix} = 15, \quad \alpha_2 = 15$$

$$\alpha_3 = \frac{\Delta\alpha_3}{\det A}, \quad \Delta\alpha_3 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 1 & 3 & 2 \end{vmatrix} = 8, \quad \alpha_3 = 8$$

$$\Rightarrow \bar{x} = -51\bar{v}_1 + 15\bar{v}_2 + 8\bar{v}_3$$

Metoda 2 (folosim metoda de trecere de la baza canonică la baza B)

$$B = \{(1,2,1), (2,3,3), (3,7,1)\}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ 1 & 3 & 1 \end{pmatrix} \text{ - matricea de trecere de la baza canonică la } B$$

În baza canonică $X = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $x = (3, -1, 2)$

$$\bar{x} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \alpha_3\bar{v}_3 \Rightarrow X = C \cdot X', \quad X' = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = ?$$

C^{-1}

$$C^{-1} \cdot X = \underbrace{C^{-1} \cdot C}_{I_3} \cdot X'$$

$$C^{-1} \cdot X = X'$$

Determinăm C^{-1} după formula $C^{-1} = \frac{1}{\det C} \cdot C^*$

$$\det C = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ 1 & 3 & 1 \end{vmatrix} = 3 + 18 + 14 - (9 + 21 + 4) = 35 - 34 = 1 \neq 0 \Rightarrow$$

$\Rightarrow C$ - inversabilă $\Rightarrow \exists\, C^{-1}$

$$C^t = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 7 & 1 \end{pmatrix}, \quad C^* = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$c_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & 3 \\ 7 & 1 \end{vmatrix} = -18$

$c_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 7$

$c_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = 5$

$c_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 2 & 1 \\ 7 & 1 \end{vmatrix} = 5$

$c_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2$

$c_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = -1$

$c_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} = 3$

$c_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$

$c_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

$$C^* = \begin{pmatrix} -18 & 7 & 5 \\ 5 & -2 & -1 \\ 3 & -1 & -1 \end{pmatrix}$$

$$C^{-1} = \frac{1}{1} \begin{pmatrix} -18 & 7 & 5 \\ 5 & -2 & -1 \\ 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} -18 & 7 & 5 \\ 5 & -2 & -1 \\ 3 & -1 & -1 \end{pmatrix}$$

$$X' = C^{-1} \cdot X = \begin{pmatrix} -18 & 7 & 5 \\ 5 & -2 & -1 \\ 3 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -54 - 7 + 10 \\ 15 + 2 - 2 \\ 9 + 1 - 2 \end{pmatrix} = \begin{pmatrix} -51 \\ 15 \\ 8 \end{pmatrix} \Rightarrow$$

$\Rightarrow \alpha_1 = -51,\ \alpha_2 = 15,\ \alpha_3 = 8 \Rightarrow \bar{x} = -51\bar{v}_1 + 15\bar{v}_2 + 8\bar{v}_3$

Baza canonică:

① în $\mathbb{R}^2$, $B = \{\bar{e}_1, \bar{e}_2\}$, $\bar{e}_1 = (1,0)$, $\bar{e}_2 = (0,1)$

Ex: $\bar{v} = (3,5)$, $\bar{v}_B = ?$

$$(3,5) = 3 \cdot (1,0) + 5 \cdot (0,1) = 3\bar{e}_1 + 3\bar{e}_2$$

② in $\mathbb{R}^3$, $B=\{\bar{e}_1,\bar{e}_2,\bar{e}_3\}$, $\bar{e}_1=(1,0,0)$, $\bar{e}_2=(0,1,0)$, $\bar{e}_3=(0,0,1)$

Exp: $\bar{v}=(2,-3,7)$, $\bar{v}_B=?$

$$(2,-3,7)=2\cdot(1,0,0)-3(0,1,0)+7(0,0,1)=2\bar{e}_1-3\bar{e}_2+7\bar{e}_3$$

③ in $\mathbb{R}_2[X]$, $B=\{1,X,X^2\}$

Exp: $\bar{v}=2+3X-5X^2$, $\bar{v}_B=?$

$$2+3X-5X^2=2\cdot 1+3\cdot X-5\cdot X^2$$