① $\boxed{A=\begin{pmatrix}1&2\\-1&3\end{pmatrix},\quad B=\begin{pmatrix}-1&0\\1&2\end{pmatrix}.}$ Calculați:

a) $2A-3B$; b) $A\cdot B$ c) A^2-B^2 d) $\det(A\cdot B)$ e) A^{-1}

a) $2A-3B=\begin{pmatrix}2&4\\-2&6\end{pmatrix}-\begin{pmatrix}-3&0\\3&6\end{pmatrix}=\begin{pmatrix}5&4\\-5&0\end{pmatrix}$

b) $A\cdot B=\begin{pmatrix}1&2\\-1&3\end{pmatrix}\cdot\begin{pmatrix}-1&0\\1&2\end{pmatrix}=\begin{pmatrix}-1+2&0+4\\1+3&0+6\end{pmatrix}=\begin{pmatrix}1&4\\4&6\end{pmatrix}$

c) $B^2-A^2=?$

$(B-A)(B+A)=\left[\begin{pmatrix}-1&0\\1&2\end{pmatrix}-\begin{pmatrix}1&2\\-1&3\end{pmatrix}\right]\cdot\left[\begin{pmatrix}-1&0\\1&2\end{pmatrix}+\begin{pmatrix}1&2\\-1&3\end{pmatrix}\right]$

$=\begin{pmatrix}-2&-2\\2&-1\end{pmatrix}\cdot\begin{pmatrix}0&2\\0&5\end{pmatrix}=\begin{pmatrix}0&-14\\0&-1\end{pmatrix}$

d) $\det(A\cdot B)=?$ $\qquad\det(A\cdot B)=\det A\cdot\det B$

$\det B=\begin{vmatrix}-1&0\\1&2\end{vmatrix}=-2-0=-2$

$\det A=\begin{vmatrix}1&2\\-1&3\end{vmatrix}=3+2=5$

$\Rightarrow \det(A\cdot B)=(-2)5=-10$

e) $A^{-1}=?$ $\qquad A^{-1}=\frac{1}{\det A}\cdot A^{*}$

$\det A=5\neq 0 \Rightarrow A$ - inversabilă $\Rightarrow \exists A^{-1}$ → există

$A^{t}=\begin{pmatrix}1&-1\\2&3\end{pmatrix}$, $\qquad A^{*}=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$

A^t ↓ transpusa $\qquad A^*$ ↓ adjuncta matricei A

$(-1)^{par}=+1$

$(-1)^{impar}=-1$

$a_{11}=\underbrace{(-1)^{1+1}}_{+}\cdot 3=3$

a_{ij}: i → linie, j → col

$a_{12} = (-1)^{1+2} \cdot 2 = -2$

$a_{22} = (-1)^{2+2} \cdot 1 = 1$

$a_{21} = (-1)^{2+1} \cdot (-1) = 1$

$\Rightarrow A^* = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{\det A} \cdot A^* = \frac{1}{5} \cdot \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}$

(2) $\boxed{A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{Calculați: a) } A^2 \text{ b) } \det A \text{ c) } A^{-1}}$

a) $A^2 = ?$

$A^2 = A \cdot A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1-3+2 & -1-1+4 & 2+1+4 \\ 3+3-1 & -3+1-2 & 6-1-2 \\ 1+6+2 & -1+2+4 & 2-2+4 \end{pmatrix}$

$= \begin{pmatrix} 0 & 2 & 7 \\ 5 & -4 & 3 \\ 9 & 5 & 4 \end{pmatrix}$

b) $\det A = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 2+12+1-(\cancel{2}-\cancel{2}-6)$

$\begin{matrix} 1 & -1 & 2 \\ 3 & 1 & -1 \end{matrix}$

$= 15+6 = 21 \neq 0$

c) $A^{-1} = ?$

$\det A = 21 \neq 0 \Rightarrow \exists A^{-1}$ $\quad A^{-1} = \frac{1}{\det A} \cdot A^*$

$A^t = \begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 2 \\ 2 & -1 & 2 \end{pmatrix}, \quad A^* = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$a_{11} = + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 4$

$a_{12} = - \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = -(-2-4) = 6$

$a_{13} = + \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = 1-2 = -1$

$a_{21} = - \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = -(6+1) = -7$

$a_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0 \qquad a_{23} = -\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -(-1-6) = 7, \quad a_{31} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$

$a_{32} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -(2+1) = -3 \qquad a_{33} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$

$$A^{-1} = \frac{1}{21}\begin{pmatrix} 4 & 6 & -1 \\ -7 & 0 & 7 \\ 5 & -3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{4}{21} & \frac{2}{7} & \frac{-1}{21} \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{5}{21} & -\frac{1}{7} & \frac{4}{21} \end{pmatrix}$$

(3) $A = \begin{pmatrix} \boxed{1} & -1 & 2 \\ 2 & -2 & 4 \\ 0 & 3 & -1 \end{pmatrix}$ $r = ?$ rangul

met 1

$\Delta_1 = |1| \neq 0 \Rightarrow r = 1$

$\Delta_2 = \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 0, \quad \Delta_3 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \quad \Delta_4 = \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = 3 \neq 0 \Rightarrow$

$\Rightarrow r = 2$

$\Delta_5 = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 0 & 3 & -1 \end{vmatrix} = 0 \Rightarrow \boxed{r = 2}$

met 2

$\det A = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 0 & 3 & -1 \end{vmatrix} = 0. \Rightarrow r < 3$

$d_1 = \begin{vmatrix} 2 & -2 \\ 0 & 3 \end{vmatrix} = 6 \neq 0 \Rightarrow r = 2$

(4) $A = \begin{pmatrix} 1 & m & 2 \\ 2 & -1 & m-1 \\ 1 & 1 & 3 \end{pmatrix}$ $m = ?$ a.î rangul să fie 3.

$\det A = \begin{vmatrix} 1 & m & 2 \\ 2 & -1 & m-1 \\ 1 & 1 & 3 \end{vmatrix} = -3 + 4 + m(m-1) - (-2 + 6m + m - 1)$

$= 1 + m^2 - m - (7m - 3) =$

$= 1+m^2-m-7m+3 = m^2-8m+4$

Pt ca $r=2$ trebuie ca $\det A \neq 0$

$m^2-8m+4=0$

$\Delta = b^2-4ac = (-8)^2-4\cdot1\cdot4 = 64-16=48$

$\sqrt{\Delta} = 4\sqrt{3}$

$$\begin{array}{r|l} 48 & 2 \\ 24 & 2 \\ 12 & 2 \\ 6 & 2 \\ 3 & 3 \\ 1 & \end{array}$$

$m_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{8-4\sqrt{3}}{2} = \frac{\cancel{2}(4-2\sqrt{3})}{\cancel{2}} = 4-2\sqrt{3}$

$m_2 = 4+2\sqrt{3}$ $\Rightarrow m \in \mathbb{R} \setminus \{4 \pm 2\sqrt{3}\}$

5

$$\boxed{\begin{cases} x-y+z=2 \\ 2x+y-z=-1 \\ -x+2y-z=3 \end{cases} \qquad x, y, z = ?}$$

$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$, $\quad s=3$, $n=3$, $r=?$

$\det A = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & -1 \end{vmatrix} = -1-1+4-(-1-\cancel{2}+\cancel{2}) = -2+4+1 = 3 \neq 0$

$\Rightarrow r=3$

$\boxed{r=s=n} \Rightarrow$ sistem compatibil determinat cu soluție unică (regula lui Cramer)

$x = \frac{\Delta x}{\det A}$, $\Delta x = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 3 & 2 & -1 \end{vmatrix} = -2+3-2-(3-4-1)$
$= -1+2 = 1$

$x = \frac{1}{3}$

$y = \frac{\Delta y}{\det A}$, $\Delta y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ -1 & 3 & -1 \end{vmatrix} = 1+6+2-(1-3-4)$
$= 9+6 = 15$

$$y = \frac{15}{3} = 5$$

$$z = \frac{\Delta z}{\det A}, \quad \Delta z = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{vmatrix} = 3+8-1-(-2-2-6) = 10+10 = 20.$$

$$z = \frac{20}{3}$$

$$S = \left\{ \frac{1}{3}, 5, \frac{20}{3} \right\}$$

(6) Să se rezolve sistemul. $\begin{cases} x - 2y + z = -1 \\ -x + 2y - z = 2 \\ 2x + y - 2z = 3 \end{cases}$

$n = 3, \ m = 3, \ r = ?$

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 2 & 1 & -2 \end{pmatrix}, \quad \det A = \begin{vmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 2 & 1 & -2 \end{vmatrix} = -\cancel{4} - 1 + \cancel{4} - (\cancel{4} - 1 - \cancel{4}) = -1 + 1 = 0$$

$$\Rightarrow r < 3$$

det. principal

$$\Delta_1 = \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = 0, \quad \Delta_2 = \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0, \quad \Delta_3 = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1+4 = 5 \neq 0$$

$$\Rightarrow r = 2$$

$\left.\begin{matrix} n = 3 \\ r = 2 \end{matrix}\right| \Rightarrow n - r = 1 \Rightarrow$ avem de calculat un det. caracteristic

studiem compatibilitatea

$$\Delta_c = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 2 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 6+1-8-(-4+2+6) = -1-4 = -5 \neq 0$$

$\Downarrow$

sistem incompatibil

$\Downarrow$

nu are soluții

Dacă $\Delta_c = 0 \Rightarrow$ S. compatibil.

Dacă $\Delta_c \neq 0 \Rightarrow$ S. incompatibil.

7) Să se rezolve sistemul $\begin{cases} 2x-y+z=1 \\ -2x+y-z=-1 \\ x+3y-2z=2 \end{cases}$

$$A=\begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 1 & 3 & -2 \end{pmatrix}, \quad \det A = 0 \Rightarrow r<3$$

$$d_p = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7 \neq 0 \Rightarrow r=2$$

$$\left.\begin{matrix} s=3 \\ r=2 \end{matrix}\right| \Rightarrow s-r=1 \Rightarrow 1\Delta_c, \quad \Delta_c = \begin{vmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} = 0 \Rightarrow$$

$\Rightarrow$ S. compatibil

$$\left.\begin{matrix} n=3 \\ r=2 \end{matrix}\right| \Rightarrow n-r=1 \Rightarrow$$ sist. nedeterminat (α) cu o' nec

Not $\boxed{z=\alpha}$

$$\begin{cases} 2x-y=1-\alpha & |\cdot 3 \\ x+3y=2+2\alpha \end{cases} \Rightarrow \begin{cases} 6x-3y=3-3\alpha \\ x+3y=2+2\alpha \end{cases} \quad (+)$$

$$7x = 5-\alpha \Rightarrow \boxed{x=\frac{5-\alpha}{7}}$$

$$\frac{5-\alpha}{7}+3y=2+2\alpha$$

$$3y = 2+2\alpha-\frac{5-\alpha}{7} \Rightarrow 21y = 14+14\alpha-5+\alpha$$

$$21y = 15\alpha+9$$

$$y=\frac{15\alpha+9}{21}=\frac{\cancel{3}(5\alpha+3)}{\cancel{21}\,7}$$

$$\boxed{y=\frac{3+5\alpha}{7}}$$

$$S=\left\{\frac{5-\alpha}{7}, \frac{3+5\alpha}{7}, \alpha\right\}$$

1. Se consideră matricea $A(a)=\begin{pmatrix} a & 1 & 2a \\ a & 1 & 0 \\ 1 & 1 & -a \end{pmatrix}$ și sistemul de ecuații $\begin{cases} ax+y+2az=a+1 \\ ax+y=0 \\ x+y-az=-1 \end{cases}$, unde a este număr real.

a) Arătați că $\det(A(2))=4$.

b) Pentru $a=1$, arătați că sistemul de ecuații are o infinitate de soluții.

c) Determinați numărul real a pentru care sistemul de ecuații are soluția unică (x_0, y_0, z_0) și $x_0=a$.

b) $a=1 \Rightarrow \begin{cases} x+y+2z=2 \\ x+y=0 \\ x+y-z=-1 \end{cases}$ $\quad A=\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$\det A = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow r<3$

$d_p = \boxed{\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}} = 0-2=-2 \neq 0 \Rightarrow r=2$

$\left.\begin{matrix} s=3 \\ r=2 \end{matrix}\right| \Rightarrow s-r=1 \Rightarrow 1\ \Delta_c, \quad \Delta_c = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & -1 & -1 \end{vmatrix} = 0 \Rightarrow$ S.C.

$\left.\begin{matrix} n=3 \\ r=2 \end{matrix}\right| \Rightarrow n-r=1 \Rightarrow$ S. nedet cu ∞^1 sol.

Not $\boxed{y=\alpha}$ (nu este în d_p)

$\begin{cases} x+2z=2-\alpha \\ x=-\alpha \end{cases} \Rightarrow \begin{aligned} -\alpha+2z &= 2-\alpha \\ 2z &= 2\cancel{-\alpha+\alpha} \\ z &= 1 \end{aligned}$

$S=\{-\alpha, \alpha, 1\}$.

c) $a=?$, $a\in\mathbb{R}$ a.î. (x_0, y_0, z_0) sol unică, $x_0=a$

$\det A(a) = \begin{vmatrix} a & 1 & 2a \\ a & 1 & 0 \\ 1 & 1 & -a \end{vmatrix} = -a^2+2a^2-(2a-a^2) = a^2-2a+a^2$
$= 2a^2-2a = 2a(a-1)$

Un sistem are sol. unică dacă $\det A \neq 0$

$2a(a-1) \neq 0.$

$\underbrace{2a}_{0}\underbrace{(a-1)}_{0} = 0$ $\begin{cases} 2a = 0 \Rightarrow a = 0 \\ a-1 = 0 \Rightarrow a = 1 \end{cases}$ $\Rightarrow a \in \mathbb{R} \setminus \{0,1\}$

$$\Delta x = \begin{vmatrix} a+1 & 1 & 2a \\ 0 & 1 & 0 \\ -1 & 1 & -a \end{vmatrix} = -a(a+1) + 2a = -a^2 - a + 2a = -a^2 + a$$

$$= a(-a+1)$$

$$x = \frac{\Delta x}{\det A} = \frac{\cancel{a}\cancel{(-a+1)}^{-1}}{2\cancel{a}\cancel{(a-1)}} = -\frac{1}{2}$$

$$\left.\begin{array}{l} x_0 = -\frac{1}{2} \\ x_0 = a \end{array}\right| \Rightarrow a = -\frac{1}{2}$$