Spații vectoriale

Fie K - corp comutativ, $V \neq \emptyset$

V - spațiu vectorial peste corpul K dacă pe V se definesc două legi de compoziție:

- internă $+ : V \times V \to V$, $(\bar{v}_1, \bar{v}_2) \to \bar{v}_1 + \bar{v}_2$
- externă $\cdot : K \times V \to V$, $(\alpha, \bar{v}) \to \alpha \bar{v}$

a.î. să fie îndeplinite condițiile:

1. $(V, +)$ - grup abelian
   a) asociativitate: $(\bar{v}_1 + \bar{v}_2) + \bar{v}_3 = \bar{v}_1 + (\bar{v}_2 + \bar{v}_3)$, $\forall \bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$
   b) comutativitate: $\bar{v}_1 + \bar{v}_2 = \bar{v}_2 + \bar{v}_1$, $\forall \bar{v}_1, \bar{v}_2 \in V$
   c) element neutru: $\exists\, \bar{0} \in V$ a.î. $\bar{0} + \bar{v} = \bar{v} + \bar{0} = \bar{v}$, $\forall \bar{v} \in V$
   d) element simetrizabil: $\forall \bar{v} \in V$, $\exists (-\bar{v}) \in V$ a.î. $\bar{v} + (-\bar{v}) = (-\bar{v}) + \bar{v} = \bar{0}$
2. Proprietățile înmulțirii cu scalari
   a) $\alpha(\bar{v}_1 + \bar{v}_2) = \alpha \bar{v}_1 + \alpha \bar{v}_2$
   b) $(\alpha_1 + \alpha_2)\bar{v} = \alpha_1 \bar{v} + \alpha_2 \bar{v}$
   c) $\alpha_1(\alpha_2 \bar{v}) = (\alpha_1 \alpha_2)\bar{v} = \alpha_2(\alpha_1 \bar{v})$
   d) $1 \cdot \bar{v} = \bar{v}$, unde: $\alpha, \alpha_1, \alpha_2 \in K$, $\bar{v}, \bar{v}_1, \bar{v}_2 \in V$

Obs:
- elementele lui V se numesc vectori
- elementele lui K se numesc scalari
- dacă $K = \mathbb{R} \to$ spațiu vectorial real
  $K = \mathbb{C} \to$ spațiu vectorial complex

Ex. 1 Pe $\mathbb{R}$ se definesc operațiile $x \oplus y = x+y-2$

$\lambda \odot x = \lambda x + 2 - 2\lambda$

Să se verifice dacă $(\mathbb{R}, \oplus, \odot)$ este un spațiu vectorial peste corpul $\mathbb{R}$.

1) $(\mathbb{R}, \oplus)$ – grup abelian

a) comutativitatea : $x \oplus y = y \oplus x, \forall x,y \in \mathbb{R}$

$x \oplus y = x+y-2 = y+x-2 = y \oplus x \Rightarrow$ „$\oplus$" comutativă

b) asociativitatea: $\underbrace{(x \oplus y) \oplus z}_{M_1} = \underbrace{x \oplus (y \oplus z)}_{M_2}, \forall x,y,z \in \mathbb{R}$

$M_1 = (x \oplus y) \oplus z = (\underbrace{x+y-2}_{a}) \oplus z = a \oplus z = a+z-2 = x+y-2+z-2 =$

$= x+y+z-4$

$M_2 = x \oplus (y \oplus z) = x \oplus (\underbrace{y+z-2}_{b}) = x \oplus b = x+b-2 = x+y+z-2-2 =$

$= x+y+z-4$

$\Rightarrow M_1 = M_2 \Rightarrow$ „$\oplus$" asociativă

c) element neutru : $\exists e \in \mathbb{R}$ a.î. $e \oplus x = x \oplus e = x, \forall x \in \mathbb{R}$

$e \oplus x = x \Rightarrow e + \cancel{x} - 2 = \cancel{x} \Rightarrow e = 2$

d) element simetrizabil: $\forall x \in \mathbb{R}, \exists x' \in \mathbb{R}$ a.î. $x' \oplus x = x \oplus x' = e$

$x' \oplus x = e \Rightarrow x' + x - 2 = 2 \Rightarrow x' = 2+2-x \Rightarrow x' = 4-x \in \mathbb{R}$

Din a) – d) $\Rightarrow (\mathbb{R}, \oplus)$ – grup abelian

2) a) $\underbrace{\alpha \odot (x \oplus y)}_{M_1} = \underbrace{\alpha \odot x \oplus \alpha \odot y}_{M_2}, \forall x,y,\alpha \in \mathbb{R}$

$M_1 = \alpha \odot (x \oplus y) = \alpha \odot (\underbrace{x+y-2}_{a}) = \alpha \odot a = \alpha a + 2 - 2\alpha =$

$= \alpha(x+y-2) + 2 - 2\alpha = \alpha x + \alpha y - 2\alpha + 2 - 2\alpha = \alpha x + \alpha y - 4\alpha + 2$

$M_2 = \alpha \odot x \oplus \alpha \odot y = (\alpha x + 2 - 2\alpha) \oplus (\alpha y + 2 - 2\alpha) =$

$= \alpha x + 2 - 2\alpha + \alpha y + \cancel{2} - 2\alpha - \cancel{2} = \alpha x + \alpha y - 4\alpha + 2$

$M_1 = M_2 \Rightarrow$ „adev"

b) $\underbrace{(\alpha + \beta) \odot x}_{M_1} = \underbrace{(\alpha \odot x) \oplus (\beta \odot x)}_{M_2}, \forall \alpha, \beta, x \in \mathbb{R}$

$M_1 = (\alpha+\beta)\odot x = (\alpha+\beta)x + 2 - 2(\alpha+\beta) = \alpha x + \beta x - 2\alpha - 2\beta + 2$

$M_2 = (\alpha\odot x)\oplus(\beta\odot x) = (\underbrace{\alpha x + 2 - 2\alpha}_{a})\oplus(\underbrace{\beta x + 2 - 2\beta}_{b}) = a\oplus b =$

$= a + b - 2 = \alpha x + 2 - 2\alpha + \beta x + \cancel{2} - 2\beta - \cancel{2} = \alpha x + \beta x - 2\alpha - 2\beta + 2$

$M_1 = M_2 \Rightarrow$ „adev"

c) $\underbrace{\alpha\odot(\beta\odot x)}_{M_1} = \underbrace{(\alpha\beta)\odot x}_{M_2} = \underbrace{\beta\odot(\alpha\odot x)}_{M_3}, \quad \forall \alpha,\beta,x\in\mathbb{R}$

$M_1 = \alpha\odot(\beta\odot x) = \alpha\odot(\underbrace{\beta x + 2 - 2\beta}_{a}) = \alpha\odot a = \alpha a + 2 - 2\alpha =$

$= \alpha(\beta x + 2 - 2\beta) + 2 - 2\alpha = \alpha\beta x + \cancel{2\alpha} - 2\alpha\beta + 2 - \cancel{2\alpha} =$

$= \alpha\beta x - 2\alpha\beta + 2$

$M_2 = (\alpha\beta)\odot x = \alpha\beta x + 2 - 2\alpha\beta$

$M_3 = \beta\odot(\alpha\odot x) = \beta\odot(\underbrace{\alpha x + 2 - 2\alpha}_{b}) = \beta\odot b = \beta b + 2 - 2\beta =$

$= \beta(\alpha x + 2 - 2\alpha) + 2 - 2\beta = \beta\alpha x + \cancel{2\beta} - 2\alpha\beta + 2 - \cancel{2\beta} =$

$= \alpha\beta x - 2\alpha\beta + 2$

$M_1 = M_2 = M_3 \Rightarrow$ „adev"

d) $1\odot x = x, \quad \forall x\in\mathbb{R}$

$1\odot x = 1\cdot x + 2 - 2\cdot 1 = x + 2 - 2 = x \Rightarrow$ „adev"

Exerciții propuse

① $\mathbb{R}$, $x\oplus y = x + y + 3$, $\lambda\odot x = \lambda x + 3\lambda - 3$, $\forall x, y, \lambda\in\mathbb{R}$

$(\mathbb{R}, \oplus, \odot)$ spațiu vectorial peste $\mathbb{R}$?

② $\mathbb{R}$, $x\oplus y = \sqrt[3]{x^3 + y^3}$, $\alpha\odot x = \alpha x$, $\forall x, y, \alpha\in\mathbb{R}$

$(\mathbb{R}, \oplus, \odot)$ spațiu vectorial peste $\mathbb{R}$?

③ $\mathbb{R}^3$, $\bar{x}\oplus\bar{y} = (x_1 + y_1, x_2 + y_3, x_3 - y_2)$

$\alpha\odot\bar{x} = (\alpha x_3, \alpha x_2, \alpha x_1) \quad \forall \alpha\in\mathbb{R}, \bar{x}, \bar{y}\in\mathbb{R}^3$

$(\mathbb{R}^3, \oplus, \odot)$ spațiu vectorial peste $\mathbb{R}$?