Transformări liniare

Fie V, W – spații K vectoriale
Se numește **transformare liniară** o aplicație $T: V \to W$ care îndeplinește condițiile:

① $T(\bar{v}_1 + \bar{v}_2) = T(\bar{v}_1) + T(\bar{v}_2), \quad \forall \bar{v}_1, \bar{v}_2 \in V$

② $T(\alpha \bar{v}) = \alpha T(\bar{v}), \quad \forall \alpha \in K, v \in V$

$$\Updownarrow$$

$$T(\alpha \bar{v}_1 + \beta \bar{v}_2) = \alpha T(\bar{v}_1) + \beta T(\bar{v}_2), \quad \alpha, \beta \in K, \bar{v}_1, \bar{v}_2 \in V$$

① Dacă $T: V \to W$ este bijectivă atunci T – **izomorfism** de sp vect

② Se numește **endomorfism** al sp V, orice $T: V \to V$.
Dacă endomorfismul este bij $\Rightarrow$ T – **automorfism**.

Ex 1

Să se arate că $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x_1, x_2) = (2x_1 - x_2, 3x_1, 2x_2 - 3x_1)$ este o transformare liniară

a) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v}), \quad \forall \bar{u}, \bar{v} \in \mathbb{R}^2$

$\bar{u} = (a_1, a_2), \quad \bar{v} = (b_1, b_2)$

$T(\bar{u} + \bar{v}) = T\big((a_1, a_2) + (b_1, b_2)\big) = T(\underbrace{a_1 + b_1}_{x_1}, \underbrace{a_2 + b_2}_{x_2}) =$

$= \big(2(a_1 + b_1) - (a_2 + b_2), 3(a_1 + b_1), 2(a_2 + b_2) - 3(a_1 + b_1)\big) =$

$= (2a_1 + 2b_1 - a_2 - b_2, 3a_1 + 3b_1, 2a_2 + 2b_2 - 3a_1 - 3b_1)$ ①

$T(\bar{u}) = T(a_1, a_2) = (2a_1 - a_2, 3a_1, 2a_2 - 3a_1)$

$T(\bar{v}) = T(b_1, b_2) = (2b_1 - b_2, 3b_1, 2b_2 - 3b_1)$

$T(\bar{u}) + T(\bar{v}) = (2a_1 - a_2, 3a_1, 2a_2 - 3a_1) + (2b_1 - b_2, 3b_1, 2b_2 - 3b_1) =$

$= (2a_1 - a_2 + 2b_1 - b_2,\ 3a_1 + 3b_1,\ 2a_2 - 3a_1 + 2b_2 - 3b_1)$ (2)

Din (1) și (2) $\Rightarrow T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$

b) $T(\alpha \bar{u}) = \alpha T(\bar{u})$ //

$\bar{u} = (a_1, a_2) \Rightarrow T(\alpha \bar{u}) = T(\alpha(a_1, a_2)) = T(\alpha a_1, \alpha a_2) =$

$= (2\alpha a_1 - \alpha a_2,\ 3\alpha a_1,\ 2\alpha a_2 - 3\alpha a_1)$ (1)

$\alpha \cdot T(\bar{u}) = \alpha \cdot T(a_1, a_2) = \alpha \cdot (2a_1 - a_2,\ 3a_1,\ 2a_2 - 3a_1) =$

$= (2\alpha a_1 - \alpha a_2,\ 3\alpha a_1,\ 2\alpha a_2 - 3\alpha a_1)$ (2)

Din (1) și (2) $\Rightarrow T(\alpha \bar{u}) = \alpha T(\bar{u})$

Din (a) și (b) $\Rightarrow T$ - transformare liniară.

Exercițiu propus

Să se arate că $T: \mathbb{R}^2 \to \mathbb{R}^3$,

$T(x_1, x_2) = (2x_1 + 3x_2,\ 5x_2,\ 3x_1 - 2x_2 + 5)$ este o transformare liniară

Ex2

$T: \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$, $T(A) = A^t$ A^t - transpusa matricii A

Să se arate că T - transformare liniară.

a) $T(A+B) = T(A) + T(B)$, $\forall A, B \in \mathcal{M}_2(\mathbb{R})$ //

$A \in \mathcal{M}_2(\mathbb{R}) \Rightarrow A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $A^t = \begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix}$

$B \in \mathcal{M}_2(\mathbb{R}) \Rightarrow B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, $B^t = \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix}$

$T(A+B) = (A+B)^t = \left[\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right]^t = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}^t =$

$= \begin{pmatrix} a_1+a_2 & c_1+c_2 \\ b_1+b_2 & d_1+d_2 \end{pmatrix}$ (1)

$$T(A)=A^t=\begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix}$$
$$T(B)=B^t=\begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix}$$
$$\Rightarrow T(A)+T(B)=\begin{pmatrix} a_1+a_2 & c_1+c_2 \\ b_1+b_2 & d_1+d_2 \end{pmatrix} \quad (2)$$

Din (1) și (2) $\Rightarrow T(A+B)=T(A)+T(B)$

b) $\underline{T(\alpha A)=\alpha T(A)}$, $\alpha\in\mathbb{R}$, $A\in\mathcal{M}_2(\mathbb{R})$

$$T(\alpha A)=(\alpha A)^t=\left(\alpha\cdot\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right)^t=\begin{pmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{pmatrix}^t=\begin{pmatrix} \alpha a_1 & \alpha c_1 \\ \alpha b_1 & \alpha d_1 \end{pmatrix} \quad (1)$$

$$\alpha T(A)=\alpha\cdot A^t=\alpha\cdot\begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix}=\begin{pmatrix} \alpha a_1 & \alpha c_1 \\ \alpha b_1 & \alpha d_1 \end{pmatrix} \quad (2)$$

Din (1) și (2) $\Rightarrow$ T – transf. liniară

Fie V – spațiu K vectorial
Se numește **formă liniară** o transformare liniară
$T: V\to K$ (un vector este dus într-un scalar)

$\boxed{L(V,W)}$ – mulțimea transformărilor liniare

V, W – două K-sp. vect, $T\in L(V,W)$
$\ker T=\{\bar{v}\in V \mid T(\bar{v})=\bar{0}_W\}$ – **nucleul** lui T.
$\operatorname{Im} T=\{\bar{w}\in W \mid \exists\, \bar{v}\in V \text{ a.î. } T(\bar{v})=\bar{w}\}$ **imaginea** lui T

Ex1

$T:\mathbb{R}^3\to\mathbb{R}^4$, $T(x_1,x_2,x_3)=(-x_1+x_2+x_3,\ x_1-x_2+x_3,\ 2x_3,\ x_1-x_2)$
a) $\ker T=?$
b) $\operatorname{Im} T=?$

a) $\ker T=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid T(x_1,x_2,x_3)=(0,0,0,0)\}$

$$T(x_1,x_2,x_3)=(0,0,0,0)\Rightarrow\begin{cases} -x_1+x_2+x_3=0 \\ x_1-x_2+x_3=0 \\ 2x_3=0 & \Rightarrow x_3=0 \\ x_1-x_2=0 & \Rightarrow x_1=x_2 \overset{\text{not}}{=} \alpha. \end{cases}$$

Sol: $\{\alpha, \alpha, 0\} \Rightarrow \ker T = \{(\alpha, \alpha, 0) / \alpha \in \mathbb{R}\}$, $\dim(\ker T) = 1$

b) $\operatorname{Im} T = \{(a, b, c, d) \in \mathbb{R}^4 / \exists (x_1, x_2, x_3) \in \mathbb{R}^3$ a.î. $T(x_1, x_2, x_3) = (a, b, c, d)\}$

$T(x_1, x_2, x_3) = (-x_1 + x_2 + x_3,\ x_1 - x_2 + x_3,\ 2x_3,\ x_1 - x_2) = (a, b, c, d) \Rightarrow$

$$\begin{cases} -x_1 + x_2 + x_3 = a \\ x_1 - x_2 + x_3 = b \\ 2x_3 = c \\ x_1 - x_2 = d \end{cases} \qquad A = \left(\begin{array}{ccc|c} -1 & 1 & 1 & a \\ 1 & -1 & 1 & b \\ 0 & 0 & 2 & c \\ 1 & -1 & 0 & d \end{array}\right)$$

s - nr de ec
n - nr de necun.

$\Delta_1 = |-1| \neq 0 \Rightarrow r \geq 1$

$\Delta_2 = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0$

$\Delta_3 = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0 \Rightarrow r \geq 2$

$\Delta_4 = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2 + 0 + 0 - (-0 - 0 + 2) = 0$

$\Delta_5 = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0 - 1 + 1 - (-1 + 1 + 0) = 0$

$\Rightarrow \boxed{r = 2}$

$\left.\begin{array}{l} s = 4 \\ r = 2 \end{array}\right| \Rightarrow s - 4 = 2 \Rightarrow$ avem de calculat doi det. caracteristici

$\Delta c_1 = \begin{vmatrix} -1 & 1 & a \\ 1 & 1 & b \\ 0 & 2 & c \end{vmatrix} = -c + 2a + 0 - (0 - 2b + c)$
$= -c + 2a + 2b - c = 2a + 2b - 2c$

$\Delta c_2 = \begin{vmatrix} -1 & 1 & a \\ 1 & 1 & b \\ 1 & 0 & d \end{vmatrix} = -d + 0 + b - (a - 0 + d) = -d + b - a - d =$
$= -a + b - 2d.$

Impunem cond. ca sist. să fie compatibil $\Rightarrow \Delta c_1 = \Delta c_2 = 0$

$\Delta c_1 = 0 \Rightarrow 2a + 2b - 2c = 0 \,/ :2 \Rightarrow a + b - c = 0 \Rightarrow \boxed{c = a + b}$

$\Delta c_2 = 0 \Rightarrow -a + b - 2d = 0 \Rightarrow 2d = -a + b \Rightarrow \boxed{d = \frac{-a+b}{2}}$

$\operatorname{Im} T = \{(a, b, a+b, \frac{-a+b}{2}) / a, b \in \mathbb{R}\}$ $\qquad \dim(\operatorname{Im} T) = 2$

$$\dim(\ker T) + \dim(\operatorname{Im} T) = 2+1 = 3 = \dim(\mathbb{R}^3)$$

Ex 2

Determinați transformarea liniară $T:\mathbb{R}^3 \to \mathbb{R}^3$ cu proprietatea $T(\bar{v}_i) = \bar{w}_i$, $i=\overline{1,3}$

$\bar{v}_1 = (1,1,0)$, $\bar{v}_2 = (1,0,1)$, $\bar{v}_3 = (0,1,1)$

$\bar{w}_1 = (2,1,0)$, $\bar{w}_2 = (-1,0,1)$, $\bar{w}_3 = (1,1,1)$

$T(x_1,x_2,x_3) = ?$

a) verificăm dacă $\bar{v}_1, \bar{v}_2, \bar{v}_3$ formează o bază în $\mathbb{R}^3$

$$\bar{v} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \alpha_3\bar{v}_3$$

$$\alpha_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(0,1,1) = \bar{0}$$

$$(\alpha_1,\alpha_1,0) + (\alpha_2,0,\alpha_2) + (0,\alpha_3,\alpha_3) = (0,0,0)$$

$$\begin{cases}\alpha_1+\alpha_2 = 0\\ \alpha_1+\alpha_3 = 0\\ \alpha_2+\alpha_3 = 0\end{cases}, \quad A = \begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}, \quad \det A = \begin{vmatrix}1&1&0\\1&0&1\\0&1&1\end{vmatrix} = 0+0+0-(0+1+1) = -2 \neq 0 \quad \text{S.C.D.}$$

(sol unică Cramer)

$$\alpha_1 = \frac{\Delta\alpha_1}{\det A}, \quad \Delta\alpha_1 = \begin{vmatrix}0&1&0\\0&0&1\\0&1&1\end{vmatrix} = 0 \Rightarrow \alpha_1 = \frac{0}{-2} = 0$$

Analog $\alpha_2 = \alpha_3 = 0 \Rightarrow \bar{v}_1, \bar{v}_2, \bar{v}_3$ l.i $\Rightarrow$ formează o bază în $\mathbb{R}^3$.

$$\bar{v} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \alpha_3\bar{v}_3$$

$$(x,y,z) = (\alpha_1,\alpha_1,0) + (\alpha_2,0,\alpha_2) + (0,\alpha_3,\alpha_3) \qquad \alpha_1,\alpha_2,\alpha_3 = ?$$

$$\begin{cases}\alpha_1+\alpha_2 = x\\ \alpha_1+\alpha_3 = y\\ \alpha_2+\alpha_3 = z\end{cases}, \quad A = \left(\begin{array}{ccc|c}1&1&0&x\\1&0&1&y\\0&1&1&z\end{array}\right) \qquad \det A = -2 \neq 0 \quad \text{S.C.D.}$$

$$\alpha_1 = \frac{\Delta\alpha_1}{\det A}, \quad \Delta\alpha_1 = \begin{vmatrix}x&1&0\\y&0&1\\z&1&1\end{vmatrix} = 0+0+z-(0+x+y) = z-x-y$$

$$\alpha_1 = \frac{-x-y+z}{-2} = \frac{x+y-z}{2}$$

$$\alpha_2 = \frac{\Delta\alpha_2}{\det A}, \quad \Delta\alpha_2 = \begin{vmatrix}1&x&0\\1&y&1\\0&z&1\end{vmatrix} = y+0+0-(0+z+x) = -x+y-z$$

$$\alpha_2 = -\frac{x+y-z}{2} = \frac{x-y+z}{2}$$

$$\alpha_3 = \frac{\Delta\alpha_3}{\det A}, \quad \Delta\alpha_3 = \begin{vmatrix} 1 & 1 & x \\ 1 & 0 & y \\ 0 & 1 & z \end{vmatrix} = 0+x+0-(0+y+z) = x-y-z$$

$$\alpha_3 = \frac{x-y-z}{-2} = \frac{-x+y+z}{2}$$

$$\boxed{\bar{v} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \alpha_3\bar{v}_3} \Big/ \underset{T}{=} \frac{x+y-z}{2}\bar{v}_1 + \frac{x-y+z}{2}\bar{v}_2 + \frac{-x+y+z}{2}\bar{v}_3$$

$$T(\bar{v}) = T(\alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \alpha_3\bar{v}_3) = \alpha_1\underbrace{T(\bar{v}_1)}_{\bar{w}_1} + \alpha_2\underbrace{T(\bar{v}_2)}_{\bar{w}_2} + \alpha_3\underbrace{T(\bar{v}_3)}_{\bar{w}_3}$$

$$= \alpha_1\bar{w}_1 + \alpha_2\bar{w}_2 + \alpha_3\bar{w}_3 =$$

$$= \frac{x+y-z}{2}\cdot(2,1,0) + \frac{x-y+z}{2}\cdot(-1,0,1) + \frac{-x+y+z}{2}\cdot(1,1,1) =$$

$$= \left(x+y-z, \frac{x+y-z}{2}, 0\right) + \left(\frac{-x+y-z}{2}, 0, \frac{x-y+z}{2}\right) + \left(\frac{-x+y+z}{2}, \frac{-x+y+z}{2}, \frac{-x+y+z}{2}\right) =$$

$$= \left(x+y-z + \frac{\cancel{-x+y-z} + \cancel{-x+y+z}}{2}, \frac{\cancel{x}+y-\cancel{z}\cancel{-x}+y+\cancel{z}}{2}, \frac{\cancel{x}-\cancel{y}+z\cancel{-x}+\cancel{y}+z}{2}\right)$$

$$= (x+y-z, y, z)$$

$$T(\bar{v}) = T(x,y,z) = (x+y-z, y, z)$$

Exerciții propuse

① Fie $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x_1,x_2) = (x_1-x_2, x_1+x_2, 2x_1+3x_2)$

a) $\ker T = ?$

b) $\operatorname{Im} T = ?$

② Studiați care dintre următoarele aplicații sunt transformări liniare:

a) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x_1,x_2) = (x_1, -x_1)$

b) $T: V_3 \to \mathbb{R}$, $T(\bar{v}) = \bar{a}\cdot\bar{v}$, $\bar{a} \in V_3$ (vector fixat)

$\bar{v} = (x_1,x_2,x_3)$ și spre exp $\bar{a} = (1,-1,2)$

c) $T: \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$, $T(A) = \operatorname{Trace}(A)$, $\boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \operatorname{Trace} A = a+d}$

③ Fie $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x+y-z, 2x-y+z)$

a) Arătați că T - transf liniară

b) $\ker T = ?$

c) $\operatorname{Im} T = ?$

④ Fie $T: \mathbb{R}^4 \to \mathbb{R}^3$, $T(x,y,z,t) = (x+y+z, y-t, z-t)$

a) Arătați că T - transf liniară

b) $\ker T = ?$

c) $\operatorname{Im} T = ?$