www.profesorciofuconstantin

Diferențiala unei funcții

$$\boxed{df(x,y) = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy}$$ – diferențiala lui f

$$\boxed{\text{grad } f = \frac{\partial f}{\partial x}(x,y)\cdot\bar{i} + \frac{\partial f}{\partial y}(x,y)\cdot\bar{j}}$$ – gradientul lui f

$$\boxed{\text{grad } f = \nabla f}$$

Ex 1

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + 2xy + y^2$

a) grad f = ?

b) derivata lui f după direcția MN, unde $M(1,2)$, $N(4,6)$

a) $\text{grad } f = \frac{\partial f}{\partial x}(x,y)\cdot\bar{i} + \frac{\partial f}{\partial y}(x,y)\cdot\bar{j}$

$$\left.\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}(x^2+2xy+y^2) = 2x+2y \\ \frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}(x^2+2xy+y^2) = 2x+2y\end{aligned}\right| \Rightarrow$$

$\Rightarrow \text{grad } f = (2x+2y)\bar{i} + (2x+2y)\bar{j}$

$M(1,2) \Rightarrow \text{grad } f = 6\bar{i} + 6\bar{j} = (6,6)$ – gradientul lui f în pct $M(1,2)$

$\overrightarrow{MN} = (x_N - x_M)\cdot\bar{i} + (y_N - y_M)\cdot\bar{j} = 3\bar{i} + 4\bar{j}$

$|\overrightarrow{MN}| = \sqrt{(x_M-x_N)^2 + (y_M-y_N)^2} = \sqrt{(1-4)^2+(2-6)^2} = \sqrt{25} = 5$

$\bar{v} = \frac{\overrightarrow{MN}}{|\overrightarrow{MN}|} = \frac{3\bar{i} + 4\bar{j}}{5} = \frac{3}{5}\bar{i} + \frac{4}{5}\bar{j} \Rightarrow v\left(\frac{3}{5}, \frac{4}{5}\right)$

$\frac{\partial f}{\partial v}(M) = \underbrace{df(M)}(\bar{v}) \longrightarrow$ diferențiala lui f în M

derivata lui f în M pe direcția lui v

$\bar{v}\left(\frac{3}{5}, \frac{4}{5}\right)$

$$df(M)(v) = \frac{\partial f}{\partial x}(1,2)(v_1) + \frac{\partial f}{\partial y}(1,2)\cdot(v_2) = 6v_1 + 6v_2 =$$

$$= 6\cdot\frac{3}{5} + 6\cdot\frac{4}{5} = \frac{18}{5} + \frac{24}{5} = \frac{42}{5}$$

$$\Rightarrow \frac{\partial f}{\partial v}(M) = \frac{42}{5}$$

Exp 2

$f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 2x^3 - 8yz + 3y^2$

a) grad $f = ?$

b) derivata lui f după direcția MN, unde $M(1,1,0)$, $N(0,-1,1)$

a) $\text{grad } f = \frac{\partial f}{\partial x}(x,y,z)\cdot\bar{i} + \frac{\partial f}{\partial y}(x,y,z)\cdot\bar{j} + \frac{\partial f}{\partial z}(x,y,z)\cdot\bar{k}$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{\partial}{\partial x}\left(2x^3 - 8yz + 3y^2\right) = 6x^2$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\left(2x^3 - 8yz + 3y^2\right) = -8z + 6y$$

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}\left(2x^3 - 8yz + 3y^2\right) = -8y$$

$\Rightarrow$

$$\Rightarrow \text{grad } f = 6x^2\,\bar{i} + (-8z + 6y)\cdot\bar{j} + (-8y)\cdot\bar{k}$$

$$\frac{\partial f}{\partial x}(1,1,0) = 6\cdot 1^2 = 6$$

$$\frac{\partial f}{\partial y}(1,1,0) = -8\cdot 0 + 6\cdot 1 = 6$$

$$\frac{\partial f}{\partial z}(1,1,0) = -8\cdot 1 = -8$$

$\Rightarrow \operatorname{grad} f(1,1,0) = 6\bar{i} + 6\bar{j} - 8\bar{k} = (6,6,-8)$

gradientul lui f în pct $M(1,1,0)$

$$\overrightarrow{MN} = (x_N - x_M)\bar{i} + (y_N - y_M)\bar{j} + (z_N - z_M)\cdot\bar{k} = -\bar{i} - 2\bar{j} + \bar{k}$$

$$\overline{MN} = (-1,-2,1)$$

$$|\overrightarrow{MN}| = \sqrt{(x_M - x_N)^2 + (y_M - y_N)^2 + (z_M - z_N)^2} = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

$$\bar{v} = \frac{\overrightarrow{MN}}{|\overrightarrow{MN}|} = \frac{-\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{6}} = \frac{-1}{\sqrt{6}}\bar{i} - \frac{2}{\sqrt{6}}\bar{j} + \frac{1}{\sqrt{6}}\bar{k} \Rightarrow$$

$$\Rightarrow \bar{v}\left(-\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\frac{\partial f}{\partial v}(M) = df(M)\cdot\bar{v} = df(1,1,0)\cdot\left(\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) =$$

$$= (6,6,-8)\cdot\left(-\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{6}{\sqrt{6}} - \frac{12}{\sqrt{6}} - \frac{8}{\sqrt{6}} =$$

$$= -\frac{26}{\sqrt{6}} = -\frac{26\sqrt{6}}{6} = -\frac{13\sqrt{6}}{3}$$

Divergența

Fie $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$, $\bar{v} = (v_1, v_2, v_3)$

$$\boxed{\operatorname{div}\bar{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}}$$ – divergența unui câmp vectorial

sau

$$\boxed{\operatorname{div}\bar{v} = \nabla\cdot\bar{v}}$$ (produs scalar dintre gradient și $\bar{v}$)

unde: $\begin{cases} \nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k} \\ \bar{v} = (v_1, v_2, v_3) \end{cases}$

Rotorul

Fie $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$, $\quad \bar{v} = (v_1, v_2, v_3)$

$$\boxed{\operatorname{rot}\bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}}$$ – rotorul unui câmp vectorial

sau

$$\boxed{\operatorname{rot}\bar{v} = \nabla \times \bar{v}}$$ (produs vectorial)

Exp 3) Fie $f: \mathbb{R}^3 \to \mathbb{R}$, $f = x^2y + yz^2$ și câmpul vectorial $\bar{v}(x,y,z) = (\underbrace{x^2}_{v_1}, \underbrace{2xy}_{v_2}, \underbrace{3z}_{v_3})$

a) $\operatorname{grad} f = ?$

b) $\operatorname{div}\bar{v} = ?$

c) $\operatorname{rot}\bar{v} = ?$

a) $\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k} =$

$= 2xy\,\bar{i} + (x^2+z^2)\bar{j} + 2yz\,\bar{k}$

b) $\operatorname{div}\bar{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 2x + 2x + 3 =$

$= 4x + 3$

d) $\operatorname{rot} \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xy & 3z \end{vmatrix} =$

$= \bar{i} \cdot \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & 3z \end{vmatrix} - \bar{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2 & 3z \end{vmatrix} + \bar{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 & 2xy \end{vmatrix} =$

$= \bar{i} \left(\underbrace{\frac{\partial 3z}{\partial y}}_{0} - \underbrace{\frac{\partial 2xy}{\partial z}}_{0} \right) - \bar{j} \left(\underbrace{\frac{\partial 3z}{\partial x}}_{0} - \underbrace{\frac{\partial x^2}{\partial z}}_{0} \right) + \bar{k} \left(\underbrace{\frac{\partial 2xy}{\partial x}}_{2y} - \frac{\partial x^2}{\partial y} \right) = 2y\,\bar{k}$

$\operatorname{rot} \bar{v} = 0\bar{i} + 0\bar{j} + 2y\,\bar{k}$